EXPLICIT POSITIVE-DENSITY COLLATZ CONVERGENCE IN LOGARITHMIC TIME

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ABSTRACT. Let $T(n) = n/2$ for even n and $T(n) = 3n+1$ for odd n, and let $\tau(n)$ be the least number of iterates of T needed to reach 1. We give an explicit positive rational c and an explicit positive integer X_0 such that, for every integer $X \geq X_0$, at least cX positive integers $n < X$ satisfy $\tau(n) \leq (523/50)\log n$. Thus convergence to 1 in at most $10.46 \log n$ ordinary Collatz steps holds on a set of positive lower natural density. The argument combines finite ternary mixing with a weighted inverse-orbit construction, a positive limit obtained from one fixed convergent seed, and a counting estimate that retains the same restricted histories throughout. An estimate for the complete valuation history, including the final step from the seed to 1, gives the numerical time constant. Both c and X_0 are specified by finite arithmetic expressions, without unspecified mixing, seed or eventuality constants. The density bound is extremely small and the cutoff extremely large; no practical numerical range is claimed. The theorem is accompanied by a Lean 4 formalization.

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1. INTRODUCTION

The Collatz conjecture asserts that every positive integer eventually reaches 1 under iteration of

$$T(n) = \begin{cases} n/2, & 2 \mid n, \\ 3n+1, & 2 \nmid n. \end{cases} \tag{1.1}$$

Write $\tau(n) = \min\{t \geq 0 : T^t(n) = 1\}$ when this set is nonempty, and $\tau(n) = \infty$ otherwise. In particular, $\tau(1) = 0$. Throughout, log denotes the natural logarithm. We count every application of (1.1); an odd step and its subsequent divisions by 2 are not identified as one step.

Theorem 1.1. *The rational c and integer X_0 defined in Section 8 are positive. For every integer $X \geq X_0$,*

$$\#\{n \in \mathbb{N} : 1 \leq n < X,\ \tau(n) \leq \tfrac{523}{50}\log n\} \geq cX. \tag{1.2}$$

Consequently, the set of positive integers satisfying this time bound has positive lower natural density. Neither the existence of its natural density nor density one is asserted. The result does not settle the Collatz conjecture.

Corollary 1.2. *The same c and X_0 satisfy (1.2) with $262/25 = 10.48$ in place of $523/50 = 10.46$.*

Proof. For every $n \geq 1$, $\log n \geq 0$ and $523/50 < 262/25$. □

Krasikov and Lagarias obtained a lower bound of $X^{0.84}$ for the number of convergent positive integers at most X, for sufficiently large X [1]. More recently, Mazur [3] proved

$$\pi_a(X) \geq X^{0.90},$$

eventually for each fixed positive target a with $3 \nmid a$, and the stronger constant-factor form $\pi_a(X) \geq c_a X^{0.901}$ for some $c_a > 0$. Here $\pi_a(X)$ counts positive integers at most X whose orbit reaches a under

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the one-division map, which sends odd n to $(3n+1)/2$ and even n to $n/2$. For target $a = 1$, convergence under this map is equivalent to convergence under (1.1). Theorem 1.1 strengthens this target-1 counting conclusion from a sublinear power to a linear lower bound, and additionally restricts the ordinary total stopping time to $10.46 \log n$. It does not assert a positive-density result for every target. The predecessor exponent theorem is a comparison, not an input to the proof below.

A different type of almost-everywhere result is due to Tao: for every function tending to infinity, almost every starting value in logarithmic density has an orbit attaining a value below that function [2]. Attaining a slowly growing bound does not by itself give convergence to 1. Theorem 1.1 concerns a smaller, positive-natural-density set but supplies both convergence to 1 and a numerical ordinary-time bound. These are different conclusions; no conversion from logarithmic density to natural density is presumed.

Mazur [4] established the corresponding almost-bounded conclusion in ordinary natural density, with a uniform ordinary-time constant

$$C_{\mathrm{ND}} = \frac{1509503}{5000 \log 2} < 436.$$

Precisely, for every $f : \mathbb{N} \to \mathbb{R}$ tending to infinity, a natural-density-one set of positive integers n has an iterate $T^t(n) < f(n)$ with $t \le C_{\mathrm{ND}} \log n$. This earlier result and Theorem 1.1 are complementary: the former applies to almost every starting integer but allows a target tending to infinity; the latter reaches the fixed target 1, with the smaller numerical time coefficient, on a set of positive lower natural density. Substituting $f \equiv 1$ into the earlier theorem is not permitted.

The proof has three principal ingredients. First, averaging over finite ternary residue spaces, together with a seed-independent convergence estimate, selects *one* convergent seed with positive limiting weighted mass. Second, the restriction needed for a short orbit is preserved while the residue weight is removed and the mass is converted to a count of integers. Finally, a bound on the entire valuation history converts the number of odd steps into ordinary Collatz time. We give the construction and the estimates before assembling the theorem.

2. Odd orbits and finite transfer operators

For a positive odd integer x, put

$$a(x) = v_2(3x+1), \qquad S(x) = \frac{3x+1}{2^{a(x)}}.$$

The map S is the Syracuse map. A finite inverse history ending at an odd integer R consists of a positive odd integer x and its actual Syracuse orbit from x to R. We list its valuations from the R end outwards. Thus a word $w = (a_1, \dots, a_d)$ corresponds to the composition

$$\Phi_w = f_{a_1} \circ \cdots \circ f_{a_d}, \qquad f_a(z) = \frac{3z+1}{2^a}, \qquad \Phi_w(x) = R.$$

This convention is useful for building inverse histories. In the forward orbit from x, the valuations occur in the reverse order.

Let $A(w) = a_1 + \cdots + a_d$ and assign the word the weight

$$\omega(w) = 3^d 2^{-A(w)}. \tag{2.1}$$

Only integral, positive, odd inverse orbits with the stated valuations are admitted. A congruence alone is not a substitute for this requirement. All intermediate sources used to continue the construction are also required to be units modulo 3.

For $q \ge 0$, let $G_q = \mathbb{Z}/3^q\mathbb{Z}$, equipped with the uniform probability measure, and write $\langle f \rangle_q = 3^{-q} \sum_{y \in G_q} f(y)$. Let μ_q be the distribution on G_q defined recursively by

$$Y_0 = 0, \qquad Y_{q+1} = 2^{-A_{q+1}}(3Y_q + 1) \pmod{3^{q+1}}, \qquad \mathbb{P}(A_j = a) = 2^{-a} \quad (a \ge 1),$$

where the A_j are independent. Define

$$\rho_q(y) = \tfrac{2}{3}\, 3^q \mu_q(y). \tag{2.2}$$

Then $\rho_q \ge 0$, $\langle \rho_q \rangle_q = 2/3$, and, for $q \ge 1$, ρ_q vanishes on nonunits. The factor $2/3$ occurs once in this normalization, not once per generation.

The fine-scale mixing estimate for these finite distributions states that, for each positive integer A, there is $C_A \ge 0$ such that

$$\big\langle |\rho_q - \rho_m \circ \pi_{q,m}| \big\rangle_q \le \frac{2}{3} C_A m^{-A} \qquad (1 \le m \le q), \tag{2.3}$$

where $\pi_{q,m} : G_q \to G_m$ is reduction. Here C_A bounds the ℓ^1 distance between the probability law μ_q and the uniform lift of μ_m to G_q; the factor $2/3$ comes from (2.2). This is the finite-distribution form of the fine-scale estimate in Tao's analysis [2, Proposition 1.14]. We use fixed exponents 6 and 2, not an assertion of independence of the valuations along a particular integer orbit.

For a word w of length d, the affine map Φ_w induces an injection $F_w : G_t \to G_{t+d}$. More explicitly,

$$F_w(z) = 3^d 2^{-A(w)} z + \sum_{j=1}^{d} 3^{j-1} 2^{-(a_1+\cdots+a_j)} \pmod{3^{t+d}}.$$

The first term makes this independent of the lift of z. Define its weighted transfer by

$$(\mathcal{T}_w g)(y) = 3^d 2^{-A(w)} \sum_{z \in G_t : F_w(z) = y} g(z). \tag{2.4}$$

Lemma 2.1. *For every real-valued g on G_t,*

$$\langle |\mathcal{T}_w g| \rangle_{t+d} = 2^{-A(w)} \langle |g| \rangle_t.$$

The same equality holds without absolute values.

Proof. The map F_w is injective: the difference of two images is a unit times $3^d(z - z')$ modulo 3^{t+d}. Each nonempty fiber therefore contains one point. Sum (2.4) over the image and divide by 3^{t+d}. The factor 3^d cancels exactly. □

When words have different lengths, their transfers must first be placed on a common residue space. If g is defined on G_k and $Q \geq d + k$, write

$$\mathcal{T}_{w,Q} g = \mathcal{T}_w(g \circ \pi_{Q-d,k}) \quad \text{on } G_Q.$$

Lemma 2.1 then has right-hand side $2^{-A(w)} \langle |g| \rangle_k$, independently of Q.

For the positive odd endpoints and bounded words used below, these transfers agree with the actual inverse-orbit sums. Indeed, a nonempty residue fiber forces the successive divisibilities in $R_{i+1} = (2^{a_{i+1}} R_i - 1)/3$, starting at $R_0 = R$. Each resulting integer is positive and odd, so $v_2(3R_{i+1} + 1) = a_{i+1}$ exactly. The source is unique and has the residue in that fiber. If $k \geq 1$ and g vanishes on nonunits, the source is restricted to units as well; all earlier endpoints of such a Syracuse orbit are units automatically. Thus evaluating $\mathcal{T}_{w,Q} g$ at $R \bmod 3^Q$ gives its admitted source weight. Indicators depending on the word may be inserted without changing (2.1).

3. A family of inverse histories

For an integer $b \geq 200$, set

$$B(j) = \lceil j \log_2 3 \rceil, \quad w_b = \lfloor 3b/5 \rfloor, \quad \ell_b = b - w_b, \quad h_b = b + w_b,$$

and

$$e_b = \lfloor b/100 \rfloor, \qquad r_b = 2w_b - B(w_b) - 2e_b.$$

All these parameters are integers; in particular $r_b \geq 0$. For $0 \leq u \leq 2r_b$ define the barrier

$$H_{b,u}(s) = 2b + B((s-b)_+) - B((b-s)_+) + u - r_b. \tag{3.1}$$

Here $(v)_+ = \max(v, 0)$. The elementary bounds

$$3^j \leq 2^{B(j)} \leq 2 \cdot 3^j \tag{3.2}$$

keep every rounding loss explicit.

A selected word has length s equal to the first crossing of (3.1) after the lower endpoint ℓ_b. The partial sum at ℓ_b must be below the barrier, and the crossing must occur by h_b. With overshoot allowance K, we also require

$$H_{b,u}(s) \leq A(w) \leq H_{b,u}(s) + K. \tag{3.3}$$

Thus the lower-endpoint miss is part of the definition, including when the first crossing is at the earliest eligible depth. Let $\mathcal{W}(b, u, K)$ denote this finite set of words.

Fix an initial scale $b_0 \geq 32^5$ and put

$$b_{j+1} = b_j + \lfloor b_j/100 \rfloor, \qquad K_j = \lfloor j/100 \rfloor, \qquad m_b = \lceil b^{3/5} \rceil. \tag{3.4}$$

Starting from an odd $M \geq 16^{b_0}$, successively append admitted inverse words in $\mathcal{W}(b_j, r_{b_j}, K_j)$. Every history retains its entire word and all its intermediate endpoints. Distinct histories are not identified

merely because their last integers agree. Multiply weights when words are appended. A history is called *central* if each appended word at scale b_j has length in $[b_j - m_{b_j}, b_j + m_{b_j}]$.

For a selected word at shift r_b, the affine formula and (3.2) give the useful physical bounds

$$\tfrac{1}{4}(4/3)^b R \le R' \le 2^{K+1}(4/3)^b R. \tag{3.5}$$

Here the affine offset is at most 2^b, whereas $R \ge 16^b$; subtracting that offset loses at most a factor 2. The other factor 2 comes from (3.2). Since $(4/3)^5 > 4$, the lower bound in (3.5) is at least $16^{\lfloor b/100 \rfloor} R$ for $b \ge 200$. In particular each new endpoint is at least $16^{b_{j+1}}$, so the construction can be iterated indefinitely. The central condition is an indicator on these histories, not a separately normalized stochastic process.

3.1. **Reference probabilities and variation.** Let p_j be the sum of $2^{-A(w)}$ over the central words in $\mathcal{W}(b_j, r_{b_j}, K_j)$. The geometric law and first-crossing rule give

$$0 < p_j \le 1, \qquad 1 - p_j \le (4b_j + 4) \exp\left(-\frac{m_{b_j}^2}{576 b_j}\right) + 2^{-(K_j+1)}. \tag{3.6}$$

For completeness, the centered geometric partial sums obey

$$\mathbb{P}\{|A_1 + \cdots + A_t - 2t| > v\} \le 2\exp\bigl(-\min\{v^2/(32t), v/8\}\bigr).$$

An early central crossing or a late miss forces a deviation exceeding $m_b/3$. Apply this estimate at the possible prefix depths, all at most $2b$, to obtain the exponential term in (3.6). Conditional on the predecessor of a first crossing, geometric memorylessness makes the fraction removed by the overshoot condition exactly $2^{-(K+1)}$. This proves the stated bound. Positivity follows also directly by choosing a word crossing at the central depth with zero overshoot: the word $(1, 2, \ldots, 2, 3)$ of length b has this property at shift $u = r_b$. The displayed concentration bound is used for integers $t \ge 1$ and real $v \ge 0$.

Here is the finite-transfer estimate used below. Fix integers $h \ge 0$ and $k \ge 1$. Let $\mathcal{V}$ be any finite prefix-disjoint family of words of length at most h, let $p(\mathcal{V}) = \sum_{w \in \mathcal{V}} 2^{-A(w)}$, and set $Q = h + k$. For $1 \le \ell \le Q$, with the same constant C_A,

$$\left\langle \left| \sum_{w \in \mathcal{V}} \mathcal{T}_{w,Q} \rho_k - \rho_\ell \circ \pi_{Q,\ell} \right| \right\rangle_Q \le \frac{2}{3}\left(\frac{C_A}{k^A} + \frac{C_A}{\ell^A} + 1 - p(\mathcal{V})\right). \tag{3.7}$$

To prove it, first replace each lifted ρ_k by $\rho_{Q-|w|}$. Mixing and Lemma 2.1 bound the total replacement error, since the disjoint prefix probabilities sum to at most 1. The independent geometric construction of ρ_Q partitions into these prefix events and their complement. Hence the replaced sum is pointwise at most ρ_Q, with missing mean $(2/3)(1 - p(\mathcal{V}))$. Finally compare ρ_Q with the lift of ρ_ℓ using (2.3). This proves (3.7) without assuming an independent valuation law for any fixed integer orbit.

Pair the weighted central histories at generation n with ρ_{k_n} at their current endpoints, where $k_n = \lfloor b_n/4 \rfloor$. Denote the resulting nonnegative scalar by $Z_n(M)$. Its definition uses the initial weight 1; it does not divide by the surviving mass.

Proposition 3.1 (Uniform variation)**.** *There is a summable nonnegative sequence E_n, depending on b_0 but not on the integer M, such that*

$$|Z_{n+1}(M) - Z_n(M)| \le E_n.$$

In particular, $Z_n(M)$ converges to a nonnegative limit $z(M)$ and

$$|z(M) - Z_N(M)| \le R_N := \sum_{j \ge N} E_j, \qquad R_N \longrightarrow 0. \tag{3.8}$$

Proof. We give the residue multiplicity bound explicitly, since its uniformity in M is essential. Let $H_{n,q}(y)$ be the sum of the weights of the generation-n central histories whose current endpoint is y mod 3^q. Remove the final word when comparing successive generations; the indicator of all earlier central restrictions is unchanged. On the common space G_Q, with $Q = h_{b_n} + k_{n+1}$, this expresses $Z_{n+1} - Z_n$ as the pairing of $H_{n,Q}$ with the error in (3.7), using $k = k_{n+1}$ and $\ell = k_n$.

To bound the resulting pairing at a fixed integer endpoint, group histories by total depth D and total valuation A. If $B_* = \sum_{j<n} b_j$, $W_* = \sum_{j<n} m_{b_j}$ and $K_* = \sum_{j<n} K_j$, their possible pairs satisfy

$$B_* - W_* \le D \le B_* + W_*,$$
$$2B_* + B(D) - B(B_*) - 2n \le A \le 2B_* + B(D) - B(B_*) + K_* + 2n.$$

There are at most $(2W_* + 1)(K_* + 4n + 1)$ such pairs. In a fixed pair the weight $W = 3^D 2^{-A}$ is constant. Write the affine identity as $WR + \beta = M$. A single block at scale b has offset at most 2^b, because its

length is at most $8b/5$ and $(3/2)^8 < 2^5$. Its weight is at most $16^{-\lfloor b/100\rfloor}$: the physical inverse source is at least $16^{\lfloor b/100\rfloor}$ times its parent. Induction under composition therefore gives

$$0 \le \beta \le 2^{b_0+1}, \qquad W16^{b_n} \le 16^{b_0}.$$

For the offset induction, the next-scale tail contributes at most $16^{-e}2^{b+e+1} \le 2^b$ when $e = \lfloor b/100\rfloor \ge 1$. Thus all WR in a fixed pair lie in an interval of length 2^{b_0+1}. Equal endpoints at fixed depth have the same actual valuation word, hence the same first-crossing cuts. Counting distinct endpoints in one residue class, whose spacing is at least 3^q, now gives

$$3^q H_{n,q}(y) \le (2W_* + 1)(K_* + 4n + 1)\left(2^{b_0+1} + 16^{b_0}(9/16)^{b_n}\right), \qquad q \le 2b_n. \tag{3.9}$$

The restriction on q is necessary here: it bounds the endpoint term $3^q W$ by $16^{b_0}(9/16)^{b_n}$. Consequently, for any real function f on G_q with $q \le 2b_n$,

$$\left|\sum_y H_{n,q}(y) f(y)\right| \le C_{b_0}(n+1)^3 G^n \langle |f| \rangle_q, \qquad G = \frac{2013}{2000}, \tag{3.10}$$

where one may take $C_{b_0} = 6(4b_0^{3/5} + 1)(2^{b_0+1} + 16^{b_0})$. Indeed $W_* \le 2nb_0^{3/5}G^n$ and $K_* \le n(n+1)$ suffice. Both moduli actually needed satisfy the restriction:

$$h_{b_n} + k_{n+1} \le \left(\frac{8}{5} + \frac{101}{400}\right) b_n < 2b_n, \qquad h_{b_n} + k_n \le \frac{37}{20} b_n < 2b_n.$$

Now $b_n \ge b_0(201/200)^n$ and $(101/100)^{3/5} \le G$. Also

$$(4b+4)e^{-m_b^2/(576b)} \le \frac{8 \cdot 35! \cdot 576^{35}}{b^6};$$

this follows from the 35th term of the exponential series and $m_b \ge b^{3/5}$. Put $C_{\mathrm{s}} = 8 \cdot 35! \cdot 576^{35}$. Since $k_n, k_{n+1} \ge b_n/8$, equations (3.7), (3.6), and (3.10) give the explicit majorant

$$E_n = \frac{2}{3} C_{b_0}(n+1)^3 \left(\frac{2C_6\, 8^6 + C_{\mathrm{s}}}{b_0^6} q_1^n + q_2^n\right),$$

where

$$q_1 = G(200/201)^6 < 1, \quad q_2 = G2^{-1/100} < 1. \tag{3.11}$$

This is summable. Summing increments proves (3.8). □

4. One convergent seed with positive limiting mass

Uniformity in Proposition 3.1 permits the seed to be chosen *after* fixing a tail bound, but *before* taking any limit or varying a counting cutoff.

Lemma 4.1. *Every residue class modulo 3^q contains arbitrarily large positive odd integers M with $S(M) = 1$.*

Proof. Set $R_j = (4^j - 1)/3$. For $j \ge 1$, R_j is odd and $3R_j + 1 = 4^j$, so $S(R_j) = 1$. The lifting-the-exponent identity gives

$$v_3(4^j - 1) = 1 + v_3(j), \qquad R_{j+t} = R_j + 4^j R_t.$$

Consequently $R_{j+t} \equiv R_j \pmod{3^q}$ if and only if $3^q \mid t$. The values $R_0, \dots, R_{3^q-1}$ therefore permute the residue classes modulo 3^q. Adding arbitrarily large multiples of 3^q to the index gives arbitrarily large representatives. □

Proposition 4.2. *For every $b_0 \ge 32^5$, there is a fixed odd integer $M \ge 16^{b_0}$, not divisible by 3, such that $S(M) = 1$ and $z(M) > 0$.*

Proof. By (3.6) and geometric growth of b_j, $\sum_j (1 - p_j) < \infty$. Since every $p_j > 0$, there is $p > 0$ with

$$\prod_{j<N} p_j \ge p \qquad \text{for all } N.$$

For a fixed N, composing the transfers in (2.4) yields a nonnegative function on a finite ternary residue space whose value at an integer endpoint is $Z_N(M)$. Its uniform mean is exactly

$$\frac{2}{3} \prod_{j<N} p_j. \tag{4.1}$$

Indeed, Lemma 2.1 removes one word at a time, and the final density has mean $2/3$. This function vanishes on nonunits.

Choose N so large that $R_N < p/3$, using Proposition 3.1. Some unit residue has value at least $2p/3$ by (4.1). Lemma 4.1 supplies a representative $M \geq 16^{b_0}$ of that residue with $S(M) = 1$. For this one fixed integer,

$$z(M) \geq Z_N(M) - R_N > p/3 > 0.$$

All subsequent generations use this same M. □

5. Terminal selection and removal of the residue weight

For an endpoint R at scale b, define

$$L_b(R) = \frac{4^b R}{4\,2^{r_b}3^b}, \qquad U_b(R) = 2^{2r_b} L_b(R).$$

The interval $(L_b(R), U_b(R)]$ is its range of admissible external scales. If X lies in this range, choose the least u such that $2^u L_b(R) \geq X$. Then

$$0 \leq u \leq 2r_b, \qquad X \leq 2^u L_b(R) < 2X. \tag{5.1}$$

Append a terminal word at this shift, rather than necessarily at the central shift r_b. Unlike the intermediate construction, the final integer may have any residue modulo 3.

For a terminal word of length s, impose the additional condition

$$2s \leq H_{b,u}(s) + \lfloor b/8 \rfloor. \tag{5.2}$$

This controls length without changing the word's valuation sum or its affine inverse source.

Lemma 5.1 (Loss of the terminal length restriction)**.** *Uniformly in the allowed shift u and overshoot allowance, the geometric probability of first-crossing words failing (5.2) is at most*

$$(4b+4)e^{-b/16384}. \tag{5.3}$$

Proof. The sum of the first $s-1$ valuations is below the barrier at s. This follows from the first-crossing rule, including the required miss at the lower endpoint. Failure of (5.2) consequently implies

$$A(w_{\leq s-1}) - 2(s-1) < -b/16.$$

Apply the centered geometric estimate from Section 3 at each possible predecessor depth, at most $2b$. Each probability is bounded by $2e^{-b/16384}$. The union bound proves (5.3). □

Fix the seed from Proposition 4.2. Let $\mathcal{M}_{n,X}$ be the terminal weighted sum, retaining the central condition on every earlier word, the restriction (5.2) on the last word, and the residue density ρ_{k_n} on its source. The terminal overshoot allowance here is K_n.

Proposition 5.2. *There is $a > 0$ such that $\mathcal{M}_{n,X} \geq a$ for every sufficiently large n and every X admissible for all generation-n endpoints. The lower bound and the generation threshold are independent of X.*

Proof. First omit (5.2). The terminal transfer differs from the central weighted sum $Z_n(M)$ by a quantity tending to zero, uniformly over the indicated X. To see the needed uniformity, group the endpoint-dependent shifts in (5.1) by their actual values. The ratio of any two generation-n endpoints is at most 2^{σ_n}, where $\sigma_n = \sum_{j<n}(K_j+3)$, so the selected shifts take at most $\sigma_n+1 \leq 5(n+1)^2$ values, independently of X. For a fixed shift the histogram restricted to endpoints selecting that shift is bounded pointwise by $H_{n,Q}$. Use (3.10) with $Q = h_{b_n} + k_n \leq 2b_n$ on each such histogram. For the full first-crossing family the rejection bound is

$$4e^{-b/2560000} + 2^{-(K+1)},$$

uniformly over the permitted shifts. Indeed the barrier at ℓ_b is at least $2e_b$ above its mean, and the barrier at h_b is at least $2e_b$ below its mean. Since $e_b \geq b/200$ for $b \geq 200$, the concentration estimate at these two depths gives the displayed exponential term; the same overshoot calculation supplies the other. This exponential is at most C_s/b^6, as follows from the sixth term of the exponential series and $4 \cdot 6! \cdot 2560000^6 \leq C_s$. Equation (3.7) therefore gives a bound of the form $C(n+1)^5(q_1^n + q_2^n)$, with $q_1, q_2 < 1$ as in (3.11). Here M is already fixed, so C may depend on it.

Next restrict the finite word transfers to words failing (5.2). Their sum is nonnegative; by Lemma 2.1 its uniform mean is exactly 2/3 times their geometric probability. Lemma 5.1, the identical grouping by shifts, and (3.10) therefore bound the actual residue-weighted mass removed by

$$5(n+1)^2 C_{b_0}(n+1)^3 G^n \frac{2}{3}(4b_n+4)e^{-b_n/16384},$$

which tends to zero. This uses the physical residue histogram, not only the geometric probability of the rejected words. There are three errors to budget: $|Z_n(M) - z(M)|$, the unfiltered terminal comparison, and the removed mass. Make the first at most $z(M)/4$ and each of the other two at most $z(M)/8$. Then $\mathcal{M}_{n,X} \geq z(M)/2$ uniformly in admissible X. Thus $a = z(M)/2$ is valid after enlarging the generation threshold. □

Let $\mathcal{U}_{n,X}$ be the unweighted-by-ρ terminal sum with overshoot allowance 1, retaining both the earlier central condition and (5.2). Its history weights remain (2.1). The density in $\mathcal{M}_{n,X}$ charges only unit sources; $\mathcal{U}_{n,X}$ counts every admitted source, which can only increase the right-hand side of (5.4). Write $P = M$ for the initial weight times the seed value.

Proposition 5.3 (Comparison of two residue moduli)**.** *There is $C_2 \geq 0$ such that, whenever $\mathcal{M}_{n,X} > 0$, $1 \leq m \leq k_n$ and n is sufficiently large,*

$$\mathcal{M}_{n,X} \leq \frac{8}{9}\,3^m \mathcal{U}_{n,X} + \frac{44PC_2}{m^2}. \tag{5.4}$$

Consequently $\mathcal{U}_{n,X} \geq \eta > 0$ uniformly in admissible X for every sufficiently large n.

Proof. Reduce the last valuation of a terminal word by even integers until its overshoot is 0 or 1. This valuation remains positive: before crossing, the preceding partial sum was strictly below the barrier. The earlier history and terminal depth are unchanged, as are their central and length predicates. If x is the reduced source, the source before reducing by $2j$ is

$$F_j(x) = 4^j x + \frac{4^j - 1}{3}, \qquad j \geq 0,$$

and its history weight is 4^{-j} times the reduced weight. Writing $v(z)$ for the weight of a reduced history including both indicators, this gives the upper bound

$$\mathcal{M}_{n,X} \leq \sum_z v(z) \sum_{j \geq 0} 4^{-j} \rho_{k_n}(F_j(x_z)), \qquad \sum_z v(z) = \mathcal{U}_{n,X}.$$

It is enough initially to sum a finite range of j containing all allowed overshoots; extending this nonnegative sum only increases it.

Put $k = k_n$ and $\delta = |\rho_k - \rho_m \circ \pi_{k,m}|$. Since $\rho_m \leq (2/3)3^m$ and $\sum_{j \geq 0} 4^{-j} = 4/3$, the coarse part is at most $(8/9)3^m \mathcal{U}_{n,X}$, with both indicators retained. The error, however, is evaluated at $F_j(x_z)$, not at x_z. For each j, multiplication by 4^j is invertible modulo 3^k, so F_j permutes the *full* residue group G_k and $\langle \delta \circ F_j \rangle_k = \langle \delta \rangle_k$. It need not permute the units, which is why full-group means are used.

For the error term alone, positivity permits dropping both indicators. The source-charge argument of Lemma 6.1 applies to the full reduced family: its total weight at an integer x is at most P/x. All these sources lie in $[X, 32X)$. Positivity of $\mathcal{M}_{n,X}$ supplies a parent endpoint R; using $R \geq 16^b$, $r_b \leq 2b$, and $X > L_b(R)$ gives $X > 16^b/(4\,3^b) \geq 3^b \geq 3^k$. Thus the modulus fits below X before the residue count is used. Each residue class contains at most $32X/3^k + 1 \leq 33X/3^k$ possible nonnegative integers below $32X$. Consequently, for each j,

$$\sum_z v(z)\delta(F_j(x_z)) \leq \frac{P}{X} \sum_{x \in \mathcal{S}^{\mathrm{full}}_{n,X}} \delta(F_j(x)) \leq 33P\langle \delta \rangle_k.$$

Here $\mathcal{S}^{\mathrm{full}}_{n,X}$ denotes the distinct sources of the full reduced family. Summing 4^{-j} bounds the error by

$$44P\big\langle |\rho_{k_n} - \rho_m \circ \pi_{k_n,m}| \big\rangle_{k_n}.$$

Equation (2.3) with $A = 2$ proves (5.4).

Choose an integer $m \geq 1$ such that $m^2 \geq 88PC_2/a$, where a is from Proposition 5.2. This m is fixed. For large n, $m \leq k_n$ and (5.4) gives

$$\mathcal{U}_{n,X} \geq \eta := \frac{9a}{16\,3^m} > 0. \tag{5.5}$$

□

6. From weighted histories to integer counts

Lemma 6.1 (Source charge)**.** *Let $\mathcal{S}_{n,X}$ be the set of distinct integer sources counted by $\mathcal{U}_{n,X}$. Then*

$$X\mathcal{U}_{n,X} \le P\,\#\mathcal{S}_{n,X}, \qquad \mathcal{S}_{n,X} \subset [X, 32X). \tag{6.1}$$

Proof. For a complete history of depth d, total valuation A, and source x ending at M, the affine identity gives $3^d x < 2^A M$. Hence $x\omega(w) \le M$. At a fixed generation, the full valuation word and the seed recover all central cuts, because every cut is a prescribed first crossing. The canonical orbit of a source cannot reach M twice: otherwise M would be periodic, contrary to $M > 1$ and $S(M) = 1$. Thus the weighted charge to each distinct source is bounded by P/x. This argument retains the history until its multiplicity has been accounted for. Summing and using $x \ge X$ proves the first inequality. The affine inverse formula, (3.2), and terminal overshoot at most 1 give $2^u L_b(R) \le x \le 16 \cdot 2^u L_b(R)$. Equation (5.1) therefore puts x in $[X, 32X)$. □

The choice of generation must work for every sufficiently large cutoff, not just for a sequence of geometrically separated endpoints.

Lemma 6.2 (Coverage of all large scales)**.** *For every N, there is $X_*(N)$ such that, for each $X > X_*(N)$, some $n \ge N$ has $X \in (L_{b_n}(R), U_{b_n}(R)]$ for every endpoint R of generation n.*

Proof. For a central child R' of R at scale b_j, the affine formula and (3.2) give

$$\frac{1}{4}(4/3)^{b_j} R \le R' \le 2^{K_j+1}(4/3)^{b_j} R.$$

Consequently the logarithmic spread of a generation is at most $\sigma_j = \sum_{i<j}(K_i + 3)$: the ratio of any two endpoints is at most 2^{σ_j}. This is quadratic in j, whereas b_j grows geometrically. The interval endpoint comparisons therefore eventually give

$$L_{b_{j+1}}(R') < U_{b_j}(R) \quad \text{for every generation-}(j+1)\text{ endpoint } R' \text{ and every generation-}j\text{ endpoint } R. \tag{6.2}$$

Indeed, substitute the preceding affine bounds and the definitions of L_b, U_b, r_b; the sufficient condition is $\sigma_j + K_j \le \lfloor b_j/200 \rfloor$, which holds for all large j.

Increase N until (6.2) holds thereafter, and choose $X_*(N)$ larger than every upper endpoint at generation N. If that generation is empty, its conclusion is vacuous and $n = N$ suffices. Otherwise proceed as follows. Given $X > X_*(N)$, take the first generation $n \ge N$ for which all upper endpoints are at least X. Such a generation exists, because $U_{b_j}(R) \ge 2^{b_j} \to \infty$ uniformly over its endpoints. It is later than N. Some endpoint of generation $n-1$ has upper endpoint below X. By (6.2), every lower endpoint at generation n is below that upper endpoint, hence below X. This is the required common admissible generation. □

Proposition 6.3. *Fix $C > 0$. Suppose that, for every sufficiently large generation n and every scale X admissible for that generation, every $x \in \mathcal{S}_{n,X}$ satisfies $\tau(x) \le C \log x$. Then, for every sufficiently large integer Y,*

$$\#\{1 \le x < Y : \tau(x) \le C \log x\} \ge \frac{\eta}{32P}\, Y.$$

Proof. Fix a generation threshold after which both (5.5) and the assumed source-wise time bound hold. For sufficiently large Y, Lemma 6.2 applied to $X = Y/32$ supplies an admissible generation beyond that threshold. By Lemma 6.1, its sources lie below Y and

$$\#\mathcal{S}_{n,Y/32} \ge \frac{Y}{32P}\mathcal{U}_{n,Y/32} \ge \frac{\eta}{32P} Y.$$

The seed, the residue modulus 3^m, and the positive constants were all fixed before Y; only the generation depends on Y. □

7. The ordinary-time constant

We now choose $b_0 = 512^5$. This changes none of the definitions of the central or terminal restrictions. It allows a sharper deterministic estimate for the width m_b.

Lemma 7.1. *For integers $H, b, q \ge 0$, if $H \ge 1$, $b \ge H^5$, and $2q \le H^2$, then $q\lceil b^{3/5} \rceil \le b$. In particular,*

$$100000 \sum_{j<n} m_{b_j} \le \sum_{j<n} b_j. \tag{7.1}$$

Proof. For $b \geq 1$, $\lceil b^{3/5} \rceil \leq 2b^{3/5}$. Also $b^{2/5} \geq H^2 \geq 2q$. Multiply these inequalities. Take $H = 512$ and $q = 100000$, and sum over the central generations. □

Consider a complete selected source x with n central words and one terminal word. Let d be its Syracuse depth to M, let A be the sum of these d actual valuations, and put

$$B_* = \sum_{j=0}^{n} b_j, \qquad W_* = \sum_{j<n} m_{b_j}, \qquad L = \log 2, \quad g = \log(4/3), \quad h = \log(12289/4096), \quad \gamma = 2L - h.$$

The same inverse-orbit and rounding estimates used above give

$$\log x \geq \frac{2569}{9000} B_*, \tag{7.2}$$

$$AL \geq 2dL - gW_* - (2n+1)L - \frac{b_n}{8} L. \tag{7.3}$$

To obtain (7.3), add the central-word inequalities from (3.1) and (3.2). The central depth deviation contributes at most gW_* and the ceiling losses contribute $(2n+1)L$. At the terminal crossing, (5.2) gives $2s \leq A(w) + \lfloor b_n/8 \rfloor$. For (7.2), the inverse affine formula gives, word by word,

$$\log x \geq \log M + B_* g - r_{b_n} L - 2(n+1)L.$$

The shift contributions vanish at every central word; the final word costs at most its one shift radius. From the definition of that radius and the scale recurrence,

$$10 r_{b_n} \leq 3b_n + 5, \qquad r_{b_n} \leq \frac{3}{10} b_0 + \frac{3}{1000} B_* + \frac{1}{2}.$$

Use $\log M \geq 4b_0 L$, $n + 1 \leq B_*/32^5$, $g \geq 296/1029$, and $0 < L \leq 7/10$. The initial-scale remainder is nonnegative, and the coefficient of B_* is at least

$$\frac{296}{1029} - \frac{7}{10}\left(\frac{3}{1000} + \frac{2}{32^5}\right) \geq \frac{2569}{9000}.$$

This proves (7.2) for each complete history individually.

Every odd integer before the final value 1 in these selected orbits is at least 4096. To check intermediate values, not just block endpoints, note that $S(y) \leq 2y$ for every positive odd y. If an intermediate value y has t steps remaining in a block ending at $R \geq 16^b$, then $R \leq 2^t y$. Here $t \leq h_b \leq 2(b+1)$ and $b \geq 200$, so $y \geq 2^{4b-2(b+1)} \geq 4096$. The seed has the same lower bound, and its next Syracuse value is 1. If $x = x_0, x_1, \ldots, x_d = M$ is the corresponding Syracuse orbit, then

$$AL = \log x - \log M + \sum_{j<d} \log(3 + 1/x_j) \leq \log x - \log M + dh. \tag{7.4}$$

Combining (7.3) and (7.4) yields

$$\gamma(d+1) \leq \log x - \log M + gW_* + (2n+1)L + \frac{b_n}{8} L + \gamma. \tag{7.5}$$

The extra 1 is the final Syracuse step $M \mapsto 1$.

Here are explicit bounds for all terms in (7.5):

$$W_* \leq B_*/100000, \qquad 100 b_n \leq 100 b_0 + B_*,$$

$$(2n+1)L \leq B_*/1000000, \qquad \frac{b_0}{8} L + \gamma \leq \log M.$$

The first is Lemma 7.1; the second follows by summing $b_{j+1} - b_j \leq b_j/100$; the third uses $B_* \geq 32^5(n+1)$ and $L \leq 7/10$; the last follows from $M \geq 16^{b_0}$ and $0 < \gamma \leq L$. Therefore

$$\gamma(d+1) \leq \log x + \left(\frac{g}{100000} + \frac{219}{250000}\right) B_*. \tag{7.6}$$

Lemma 7.2. *Every selected source satisfies*

$$d + 1 \leq \kappa \log x, \qquad \kappa = \frac{34881}{10000}. \tag{7.7}$$

Proof. The logarithmic series at $1/7$ and $\log(1+t) \leq t$ imply

$$g \geq \frac{296}{1029}, \qquad h \leq \log 3 + \frac{1}{12288}, \qquad \gamma \geq g - \frac{1}{12288} > 0.$$

Exact rational comparison gives

$$\frac{g}{100000}+\frac{219}{250000}\le(\kappa\gamma-1)\frac{2569}{9000}. \tag{7.8}$$

Indeed, after using $\gamma\ge g-1/12288$, the right side minus the left side is at least

$$\left(\frac{2569\kappa}{9000}-\frac{1}{100000}\right)g-\frac{2569\kappa}{9000\cdot 12288}-\frac{2569}{9000}-\frac{219}{250000}.$$

The coefficient of g is positive. Substituting $g\ge 296/1029$ therefore bounds this difference below by $604680379/126443520000000>0$. Equations (7.2), (7.6), and (7.8) give $\gamma(d+1)\le\kappa\gamma\log x$. Divide by $\gamma>0$. □

Lemma 7.3. *Every selected source satisfies $\tau(x)\le(523/50)\log x$ in the ordinary convention (1.1).*

Proof. Set $D=d+1$ and include the *entire* final valuation at M in the valuation sum A_{all}. Then $S^D(x)=1$ and $T^{D+A_{\mathrm{all}}}(x)=1$. Telescope as in (7.4), now ending at 1, to get

$$(D+A_{\mathrm{all}})L\le D(L+h)+\log x.$$

The exact inequality

$$\kappa(L+h)+1\le\frac{523}{50}L \tag{7.9}$$

follows from $3^{200}<2^{317}$, $h\le\log 3+1/12288$, and a lower bound for $\log 2$. One completely rational check is to use $\log 2>693147180/10^9$ and $\log 3<(317/200)\log 2$ in the left-hand side. The remaining linear inequality in $\log 2$ holds with these bounds. Combining (7.7) and (7.9) and dividing by $L>0$ gives $D+A_{\mathrm{all}}\le(523/50)\log x$. □

Qualitative conclusion. Choose $b_0=512^5$ and then the fixed seed supplied by Proposition 4.2. Propositions 5.2 and 5.3 give an eventual uniform lower bound $\eta>0$ for the restricted terminal mass. Lemma 7.3 supplies its source-wise time bound. Proposition 6.3 now proves the existential form of (1.2), with $c=\eta/(32M)>0$ and an unspecified sufficiently large cutoff. □

8. An explicit density and cutoff

We now specify all constants in Theorem 1.1. A finite formula, rather than a decimal expansion of an enormous integer, is used throughout. Write $\ell(u)=\lceil\log_2 u\rceil$ for positive integers u, $v_+=\max(v,0)$, and $\sqrt[6]{u}_{\mathbb{N}}=\lfloor u^{1/6}\rfloor$. All divisions inside integer recurrences are explicitly rounded down. The division defining c is rational division.

8.1. **A fixed coefficient for finite ternary mixing.** Here is a closed coefficient C for (2.3) with exponent 6. The auxiliary quantities in the following display are positive integers:

$$A_*=6409,\qquad E_*=2^{170},\qquad L_*=2^{80},$$
$$T_*=10A_*2^{3E_*},\qquad R_*=2^{E_*}\big(T_*+3(8192+3)+1\big)+1.$$

Define $g(t)=t+10\cdot 8^{32768}(t+1)^3+1+T_*$ and put

$$P_*=g^{R_*-1}(T_*),\qquad P=P_*+1,\qquad S_*=2^{34}(P_*+1).$$

Here g^j denotes exactly j compositions, with g^0 the identity; it is not a stopping rule. Continue with

$$\begin{aligned}
D_1&=(128A_*2^{3E_*})^2+2,\\
D_{\mathrm{pt}}&=2(D_1+L_*^2+1)+4,\qquad D_{\mathrm{exp}}=2^{8192A_*2^{3E_*}},\\
D_2&=\max(D_{\mathrm{pt}},D_{\mathrm{exp}},2),\\
D_{\mathrm{sc}}&=2+P+P_*^{10}+\big(8^{32768}(1+P_*)^3\big)^3+(100S_*)^2,\\
D_{\mathrm{bad}}&=(2^{38}A_*)^2+2^{38}(P+65),\\
D_3&=\max(1,D_{\mathrm{sc}},D_{\mathrm{bad}}),\qquad D_*=\max(D_1,D_2,D_3),\\
C_*&=(32A_*D_*)^{A_*},\qquad C=2C_*20^{6409}+2+2^{481}.
\end{aligned}$$

Lemma 8.1 (Numerical mixing coefficient)**.** *For $1\le m\le q$,*

$$\big\langle|\rho_q-\rho_m\circ\pi_{q,m}|\big\rangle_q\le\frac{2C}{3m^6}.$$

Quantitative dependence of the mixing estimate. The Fourier–renewal proof of the finite-distribution mixing estimate has the following numerical specialization. At primitive Fourier exponent 6409, take the separation parameter 2^{-E_*} and the localization width L_*. The three renewal bounds are satisfied with thresholds D_1, D_2, D_3 above; their common threshold is D_*. The finite time bound used in the third case is P_*, including the initial time T_*. The resulting primitive Fourier coefficient bound is $C_* h^{-6409}$ at level h for characters not factoring through level $h-1$. These are numerical instances of the renewal and Fourier estimates underlying (2.3); the detailed estimates, including this numerical specialization, are proved in the accompanying formal development.

For clarity, the conversion from that coefficient to the coefficient used here is as follows. At levels $q \ge 2^{80}$ and $9q \le 10m \le 10q$, summing the finite Fourier slices bounds the probability-law oscillation by

$$2C_* 20^{6409} q^{-7} + 2q^{-7}.$$

The second term pays the exceptional probability once. Telescoping adjacent levels gives $(2C_* 20^{6409} + 2)m^{-6}$ for $m \ge 2^{80}$. For $m < 2^{80}$ the distance between two probability distributions is at most $2 \le 2^{481} m^{-6}$. The displayed C covers both ranges. Finally multiply by the single normalization factor $2/3$ in (2.2). In particular, the coefficient in this lemma contains no unspecified constant from the qualitative mixing theorem. □

8.2. **The arithmetic recipe.** Set $b = 2^{80}$ and, for this section, use the overshoot caps $K_j = 16 + \lfloor j/100 \rfloor$. These replace the smaller caps in (3.4); the central restriction, physical intervals, history weights and terminal restriction are unchanged. Define

$$F = 2^{467} b 16^b (C+1), \tag{8.1}$$

$$N = 20000\big(\ell(F) + 64\big), \qquad N_t = 20000\big(\ell(21F) + 96\big), \tag{8.2}$$

$$\beta(0) = b, \qquad \beta(j+1) = \beta(j) + \lfloor \beta(j)/100 \rfloor. \tag{8.3}$$

For integers $u, k, j \ge 0$ define

$$Q(u,k,0) = k, \qquad Q(u,k,j+1) = u + \lfloor 3u/5 \rfloor + Q(u + \lfloor u/100 \rfloor, k, j).$$

Put

$$q = Q\big(b, \lfloor \beta(N)/4 \rfloor, N\big), \qquad \mathcal{M} = 4^{2b+1+3^q}, \tag{8.4}$$

$$m = \sqrt[6]{88 \mathcal{M} C_{\mathbb{N}}} + 1, \qquad J = \max\big(N + N_t,\ 200(\ell(m) - 78)_+,\ 20 \cdot 10^9\big). \tag{8.5}$$

Finally define the nonnegative integer exponent

$$\begin{aligned} t(u) &= 2u - \lfloor 19u/12 \rfloor, \\ E(0) &= 0, \qquad E(j+1) = E(j) + 17 + \lfloor j/100 \rfloor + t(\beta(j)), \\ B &= E(J) + \lfloor 2\beta(J)/3 \rfloor. \end{aligned}$$

The constants in Theorem 1.1 are

$$\boxed{c = \frac{3}{256 \mathcal{M} 2^m}, \qquad X_0 = 32\big(2^B \mathcal{M} + 1\big).} \tag{8.6}$$

All quantities on the right have been defined by a fixed finite number of arithmetic operations or a specified finite iteration. In particular, no chosen convergent integer or eventuality threshold remains in the recipe. Positivity and finiteness follow directly from the definitions. The dependence is in the order $b, C \longrightarrow F \longrightarrow N \longrightarrow q \longrightarrow \mathcal{M} \longrightarrow m \longrightarrow J \longrightarrow B$; the recurrence for β depends only on b.

8.3. **A bounded seed and an explicit surviving mass.** We explain why the recipe supplies the constants claimed. All marks below are the weighted sums on the original central histories, with the terminal restriction retained when it is present.

For $b_j = \beta(j)$ and the caps just specified, the deficit estimate (3.6) has the numerical consequence

$$1 - p_j \le (2^{-23} + 2^{-16})(199/200)^j.$$

Indeed the exponential strip bound is at most $2^{457}/b_j^6$; $b \ge 2^{80}$ makes its initial coefficient at most 2^{-23}. Use $b_j \ge b(201/200)^j$, $(200/201)^6 \le 199/200$ and $2^{-1/100} \le 199/200$. Consequently, for every $s \ge 0$,

$$\prod_{j<s} p_j \ge 1 - \sum_{j<s} (1 - p_j) \ge 1 - 200(2^{-23} + 2^{-16}) \ge 255/256. \tag{8.7}$$

Fix N and q before choosing a root. Pulling back the level $\lfloor \beta(N)/4 \rfloor$ reference density through N central blocks gives a function on G_q. Lemma 2.1 shows that its full mean is $(2/3)\prod_{j<N} p_j$. It vanishes

on nonunits: each nonempty transfer is supported on the image of F_w, and $F_w(z) \equiv 2^{-a_1} \pmod 3$; the initial reference density also vanishes on nonunits. Since $q \geq 1$, exactly $2 \cdot 3^{q-1}$ of the 3^q residue classes are units. Its mean over those classes is therefore exactly $\prod_{j<N} p_j$, so at least one class has mark at least $255/256$. This averaging is on a finite residue space; it does not condition or renormalize the physical history weights.

The integers $r_v = (4^v - 1)/3$ represent every residue modulo 3^q as v runs over a period of length 3^q. To see this, induction gives $v_3(4^a - 1) = 1 + v_3(a)$ for $a > 0$, so $r_{v+a} \equiv r_v \pmod{3^q}$ exactly when $3^q \mid a$. Choose the exponent in the interval $2b + 1 \leq v < 2b + 1 + 3^q$ belonging to the favorable class. Its root $M = r_v$ satisfies

$$16^b < M < \mathcal{M}, \qquad S(M) = 1.$$

The exact residue pullback re-establishes a mark at least $255/256$ for this actual root and its actual central histories. The same fixed root will be used for every later cutoff.

For these caps the pairing and variation estimates of Sections 3–6 admit the following root-independent majorants, with $r = 9999/10000$:

$$\text{central increment at } j \leq F(j+1)^3 r^j,$$
$$\text{terminal transport error at } j \leq 21F(j+1)^5 r^j,$$
$$\text{excluded terminal-depth mass at } j \leq 21F(j+1)^5 r^j.$$

The last two are different errors and must both be paid. To track the histogram factor directly, use $W_* \leq 2jb^{3/5}G^j$ and $K_* \leq 17j(j+1)$ in (3.9). Then $K_* + 4j + 1 \leq 22(j+1)^2$, so the degree-three coefficient is at most $22(4b^{3/5} + 1)(2^{b+1} + 16^b)$. Using $b^{3/5} \leq b$ and $2^{b+1} \leq 16^b$, and including the normalization $2/3$, bounds it by $220b16^b$. The strip coefficient is at most 2^{457}, so the bracket containing it, both mixing errors and overshoot failure is at most $2^{459}(C+1)$. Multiplying this bracket bound by $220b16^b$ gives at most F. The cap inequality $K_j \leq 17(j+1)$ contributes the factor $21(j+1)^2$ for terminal variation. This estimate uses the enlarged caps directly; it does not reuse the smaller-cap numerical coefficient in (3.10).

Put $s = 19999/20000$, so $r \leq s^2$ and $s^{20000} \leq 1/2$. The elementary geometric moment bound gives

$$\sum_{j \geq n} (j+1)^d r^j \leq d!\, 20000^{d+1} s^n.$$

It follows that the total central variation after N is at most $6 \cdot 20000^4/2^{64} \leq 1/16$. At every $n \geq N_t$, each terminal error is at most $120 \cdot 20000^6/2^{96} \leq 1/8$. The central debit bounds $|Z_n(M) - Z_N(M)|$ for every $n \geq N$, so no appeal to the limit $z(M)$ is needed here. Write $\mathcal{M}_{n,X}$ for the marked mass of Section 5, now using this fixed seed and the enlarged caps. At every $n \geq N + N_t$ and every admissible scale it satisfies

$$\mathcal{M}_{n,X} \geq 255/256 - 1/16 - 1/8 - 1/8 = 175/256 \geq 2/3. \tag{8.8}$$

All thresholds in this calculation depend on the fixed floor, not on the particular root selected inside its finite bound.

8.4. **Removing the residue weight with a fixed modulus.** The full reference law satisfies

$$\mu_m(y) \leq (2/3)^m, \qquad \rho_m(y) \leq (2/3)2^m. \tag{8.9}$$

For a fixed last exponent a, the map $z \mapsto 2^{-a}(3z+1)$ from G_j to G_{j+1} is injective. If y is in its image, reduction modulo 3 determines the parity of a. The sum of 2^{-a} over the eligible parity is at most $2/3$. Hence an upper bound H for atoms at level j gives an upper bound $(2/3)H$ at level $j+1$. Start with the probability-one atom at level zero. Nonnegative summability justifies interchanging the exponent sum with the finite residue sum. This is a bound for the full distribution, not a law of the selected integer histories.

For the fixed m of (8.5), use (8.9) in the same masked census as Proposition 5.3. Its upper bound for the coarse term becomes $(8/9)2^m\mathcal{U}_{n,X}$; the error remains $44P\langle|\rho_{k_n} - \rho_m \circ \pi_{k_n,m}|\rangle_{k_n}$. Both the central indicator and the terminal restriction stay in the coarse term. They may be dropped only from the nonnegative error bound. Here $P = M < \mathcal{M}$. From Lemma 8.1 and $m^6 \geq 88\mathcal{M}C$, this error is at most

$$44\mathcal{M}\frac{2C}{3m^6} \leq \frac{1}{3}.$$

Combining with (8.8) gives

$$\mathcal{U}_{n,X} \geq \frac{3}{8 \cdot 2^m}. \tag{8.10}$$

The fixed modulus really is available: $\beta(n) \geq b2^j$ when $n \geq 200j$, since $(201/200)^{200} \geq 2$. Thus $n \geq 200(\ell(m) - 78)_+$ guarantees $m \leq \lfloor \beta(n)/4 \rfloor$. All residue moduli remain powers of 3; the power 2^m in (8.10) is a density-height bound, not a replacement modulus. The physical room needed by the census follows from the positive restricted mark as in Proposition 5.3.

8.5. **An upper height for every history.** The all-history bound (3.5) retains its divisor 3^u. The elementary inequality $2^{\lfloor 19u/12 \rfloor} \leq 3^u$ gives

$$R' \leq 2^{K+1+t(u)} R.$$

Induction proves $R \leq 2^{E(j)}\mathcal{M}$ for every endpoint at generation j, without imposing the central restriction. The definitions of the physical interval and the shift radius also give, for $u \geq 200$,

$$r_u + t(u) \leq \lfloor 2u/3 \rfloor, \qquad U_u(R) \leq 2^{\lfloor 2u/3 \rfloor} R.$$

The latter bound follows by retaining the positive denominator $4 \cdot 2^{r_u} 3^u$ in $U_u(R) = 2^{2r_u} 4^u R/(4 \cdot 2^{r_u} 3^u)$. Therefore every upper interval endpoint at generation J is at most $2^B\mathcal{M}$. It is essential here to bound all histories, not just those contributing to the lower mass estimate.

The scale-overlap argument of Lemma 6.2 is effective with these caps. For a singleton root and $K_j \leq 17(j+1)$, the spread and cap are bounded by a quadratic in j, whereas the floor grows at least as $(201/200)^j$. The sufficient overlap inequality holds at every $j \geq 20 \cdot 10^9$. Explicitly, $\sigma_j + K_j \leq 20(j+1)^2$, whereas $b \geq 400$ gives

$$\left\lfloor \frac{\beta(j)}{200} \right\rfloor \geq (201/200)^j \geq \frac{\binom{j}{4}}{200^4} \geq \frac{j^4}{384 \cdot 200^4} \geq 20(j+1)^2.$$

The last two inequalities use $j \geq 8$ and $j^2 \geq 30720 \cdot 200^4$, both satisfied at the stated threshold. Hence the proof of that lemma works with any scale exceeding the maximum upper endpoint at generation J, without a further unspecified threshold.

Proof of Theorem 1.1. Use the fixed constants (8.6) and the bounded seed constructed above. Given a natural $Y \geq X_0$, put $X = Y/32$. Then $X > 2^B\mathcal{M}$. Coverage supplies a common admissible generation $n \geq J$. All three lower bounds in the definition of J hold, so (8.10) applies. Lemma 6.1 and $P = M < \mathcal{M}$ give at least

$$\frac{Y}{32\mathcal{M}} \frac{3}{8 \cdot 2^m} = cY$$

distinct positive sources strictly below Y. The ordinary-time argument in Section 7 remains valid at the larger initial floor $b = 2^{80} \geq 512^5$: it uses the central depth restriction and lower barrier, not a smaller overshoot cap. It includes the entire final step from M to 1. Every counted source x therefore satisfies $\tau(x) \leq (523/50) \log x$, as required. □

The finite expressions in (8.6) are an explicit certificate, not a claim of numerically usable constants. In particular, we do not assert a practical range for $X \geq X_0$, an asymptotic formula for the count, or optimality of any of the parameters.

9. Formal verification

The accompanying Lean 4 development formalizes the main theorem with the map (1.1), the natural logarithm, and the count of positive integers strictly below a natural-number cutoff. Its audited formal development includes a proof of the finite-distribution mixing estimate (2.3); that estimate is not an unproved assumption. The final theorem dependencies use propositional extensionality, classical choice, and quotient soundness, with no admitted proof, unproved project axiom, or native-evaluation axiom. The mathematical library is mathlib [5]. This statement concerns the formal theorem and its dependencies, not a mechanical verification of every sentence of the present exposition. Reproduction instructions and the correspondence between the mathematical statements and the formal declarations are provided in the accompanying theorem-specific source package.

Companion links: result page and paper; formal source and evidence.

References

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