NATURAL-DENSITY ALMOST-BOUNDED COLLATZ ORBITS IN LOGARITHMIC TIME

LECH MAZUR

Abstract. Here log denotes the natural logarithm and $\log_2$ the base-two logarithm. Let $\mathrm{Col}(n) = n/2$ for even n and $\mathrm{Col}(n) = 3n+1$ for odd n. We prove that the absolute constant

$$C_{\mathrm{Coll}} = \frac{1509503}{5000 \log 2} < 436$$

works for every function $f : \mathbb{N} \to \mathbb{R}$ tending to infinity: for each such f, a natural-density-one set of positive integers N has an iterate

$$\mathrm{Col}^m(N) < f(N), \qquad m \le C_{\mathrm{Coll}} \log N.$$

Only the clock is uniform; the density-one set may depend on f. For odd n, put $\mathrm{Syr}(n) = (3n+1)/2^{\nu_2(3n+1)}$. Under the weaker hypothesis that f tends to infinity only along odd integers, the analogous conclusion holds on a set of odd-relative natural density one with Syracuse clock $C_{\mathrm{Syr}} = 501501/(5000 \log 2) < 145$. Quantitatively, for every $0 < d < 5/143$ there is a constant C_d such that, for all integers $N_0 \ge 2$ and real $x \ge 2$, the number of odd inputs $N \le x$ for which $\mathrm{Syr}^m(N) > N_0$ throughout $m \le C_{\mathrm{Syr}} \log N$, divided by the ambient endpoint x, is at most $C_d(\log N_0)^{-d}$; the exact specialization $d = 6993/200000$ is retained from transport through natural counting. For the target $f(N) = \sqrt{N}$, a natural-density-one set admits a witness in

$$\frac{\log N}{2 \log 2} < m \le C_{\mathrm{Coll}} \log N \le 436 \log N,$$

and the first inequality holds for every raw witness below $\sqrt{N}$, not only the selected one. Thus the raw logarithmic order is sharp up to a constant. The proof combines natural-counting first-passage transport with Rhin's irrationality-measure input. The claim chain, constants, time conversion, and lower bracket have been checked in Lean 4.

Generative-AI disclosure. Generative AI played a substantial role in mathematical exploration, proof development, formalization, computation, and exposition. The agentic environments used included a custom harness, OpenAI Codex, and Anthropic Claude Code; the models used included GPT-5.5 and Fable 5. Lech Mazur designed the harness, curated and reconciled the outputs, and takes responsibility for the manuscript. The formal claims rest on the frozen Lean artifact identified below.

Frozen formal proof commit: `f386357d453ac4dcf91242b76252d88a5a729906`

1. Introduction

The raw Collatz map on the positive integers is

$$\mathrm{Col}(n) = \begin{cases} n/2, & n \equiv 0 \pmod 2, \\ 3n+1, & n \equiv 1 \pmod 2. \end{cases} \tag{1}$$

The Collatz conjecture asserts that every orbit reaches 1. Tao proved a far-reaching almost-everywhere substitute: if $f(N) \to \infty$, then $\inf_m \mathrm{Col}^m(N) < f(N)$ for almost all N in logarithmic density [15]. His proof is organized around first passage for the odd-to-odd Syracuse map and an approximate transport law between logarithmic blocks.

Date: Version 1, July 16, 2026.

2020 *Mathematics Subject Classification.* Primary 11B83; Secondary 11J82, 37P99, 68V20.

Key words and phrases. Collatz problem, Syracuse map, natural density, first passage, Diophantine approximation, formal proof, Lean.

As a precursor to the present extension, we formalized Tao's published almost-bounded-orbits theorem in Lean 4, including formal counterparts of the main reductions and estimates in Propositions 1.9, 1.11, 1.14, and 1.17, Lemma 7.7, Proposition 7.8, and the Section 3 telescope, rather than only its terminal statement [15]. The present development reuses that checked foundation and formalizes the additional natural-counting transport, the Rhin phase input, and the logarithmic-time bookkeeping. The exact precursor endpoint and module map are identified in section 8.

Tao explains that the multiplicative uncertainty $\exp(O(\sqrt{n}))$ in his trajectory approximation is the main reason for using logarithmic rather than natural density. He also explicitly anticipated a natural-density refinement: Remark 1.4 calls it plausible, while Remark 1.16 identifies fine-scale mixing of the entire random affine map as a likely missing input. Remark 7.5 separately suggests Baker-type control of his black set from Diophantine information on $\log 3/\log 2$ [15, Remarks 1.4, 1.16, and 7.5]. The Rhin estimate below is used at a different point: it compares natural-counting and logarithmic passage laws through phase discrepancy.

Natural-density descent has a substantial history. Independent late-1970s density-one results for the original problem or Hasse's generalization were obtained by Terras, Everett, Möller, Heppner, and Allouche. A refinement in Allouche's estimates implies descent below N^θ for fixed $\theta > 0.8691$[1]; this implication was later pointed out by a referee of Korec's paper, and Korec extended the range to every $\theta > \log_4 3$ [1, 2, 3, 5, 8, 12, 16]. For the one-division map $A(n) = n/2$ or $(3n+1)/2$, according to parity, Inselmann proved natural-density trajectory control up to the maximal logarithmic horizon; in particular, for every fixed $\varepsilon > 0$, $A^{\lfloor 2\log N/\log(4/3)\rfloor}(N) \leq N^\varepsilon$ for almost all N. His third revision also gives raw-Collatz and odd-to-odd Syracuse trajectory theorems through respective horizons $3\log N/\log(4/3)$ and $\log N/\log(4/3)$, and reaches every fixed power target as well as some subpolynomial targets [6, Theorems 1.9 and 1.10, Corollary 1.4, and the remark following Question 1.5]. The stated results do not cover arbitrary diverging thresholds in full generality, and that remark says that the methods seem unlikely to suffice for such a conclusion. Gonçalves, Greenfeld, and Madrid generalized Tao's almost-bounded theorem to a family of Collatz-type maps and discuss the natural-density obstruction for arbitrary growth functions [4]. A recent preprint of Kalafatelis reduces a complementary uniform-window shell-kernel problem to an unresolved primitive-frequency estimate; it does not state an unconditional density theorem [7]. Thus neither natural density nor a logarithmic clock alone is new here. The contribution is their combination with every threshold $f(N) \to \infty$, a quantitative natural-to-logarithmic passage comparison, and one clock retained through the global conclusion.

This paper upgrades the density conclusion to ordinary natural density and simultaneously keeps an explicit logarithmic time budget. The distinction is substantial. A set of logarithmic density one need not have natural density one, and an eventual hitting statement need not provide a uniform clock. Here one clock works for every threshold function satisfying the stated growth hypothesis: the quantifier order is $\exists C\, \forall f$, not $\forall f\, \exists C_f$.

Logarithmic time is also explicit in Tao's local argument. In (5.1) he sets $n_0 = \lfloor \log x/(10\log 2)\rfloor$ and proves, outside a power-small exception, that the first passage to a value at most x occurs by time n_0 [15, (5.1)–(5.2)]. Theorem 3.1 then obtains a fixed-target logarithmic-density estimate of order $O((\log N_0)^{-c})$, but packages reachability through an untimed orbit set, so the elapsed time is absent from the global statement [15, Theorem 3.1]. The new time assertion below is that the natural-counting transport, fixed-target splice, arbitrary-threshold conclusion, and Syracuse-to-raw conversion retain one explicit schedule throughout.

[1]The exact expression printed in Tao's 2022 published version and current arXiv v7 [15] conflicts with this decimal. We follow [2, p. 13].

For an odd positive integer n, write

$$\mathrm{Syr}(n) = \frac{3n+1}{2^{\nu_2(3n+1)}}. \tag{2}$$

Thus one Syracuse step includes the odd step and all immediately following halvings. The two clocks in the theorem therefore count different objects: $C_{\mathrm{Syr}} \log N$ counts odd-to-odd Syracuse steps, while $C_{\mathrm{Coll}} \log N$ counts every raw step of (1). Throughout, log is natural and $\log_2$ is base two.

Our headline result is the following.

Theorem 1.1 (Natural-density logarithmic-time theorem)**.** *Set*

$$C_{\mathrm{Syr}} = \frac{501501}{5000 \log 2}, \qquad C_{\mathrm{Coll}} = \frac{1509503}{5000 \log 2}. \tag{3}$$

Then the following statements hold.

(i) *If $f : \mathbb{N} \to \mathbb{R}$ tends to $+\infty$ along the odd integers, then the odd inputs N for which there is an $m \in \mathbb{N}$ satisfying*

$$m \le C_{\mathrm{Syr}} \log N, \qquad \mathrm{Syr}^m(N) < f(N)$$

have relative natural density one among the odd integers.

(ii) *If $f : \mathbb{N} \to \mathbb{R}$ tends to $+\infty$, then the positive integers N for which there is an $m \in \mathbb{N}$ satisfying*

$$m \le C_{\mathrm{Coll}} \log N, \qquad \mathrm{Col}^m(N) < f(N)$$

have natural density one.

Moreover,

$$C_{\mathrm{Syr}} < 145, \qquad C_{\mathrm{Coll}} < 436.$$

The hypothesis in part (i) is deliberately weaker: growth is required only along odd arguments. Part (ii) uses growth along all natural numbers. The conclusion is strict, $< f(N)$, not merely $\le f(N)$.

The fixed-target estimate behind theorem 1.1 retains a visible exponent.

Theorem 1.2 (Quantitative fixed-target theorem)**.** *For every real d with*

$$0 < d < \frac{5}{143}$$

there is a constant $C_d \ge 0$ such that, for all integers $N_0 \ge 2$ and all real $x \ge 2$,

$$\frac{1}{x} \#\Big\{ N \in \mathbb{N} : 1 \le N \le \lfloor x \rfloor,\ N \text{ odd}, \\ \nexists m \in \mathbb{N} :\ m \le C_{\mathrm{Syr}} \log N \text{ and } \mathrm{Syr}^m(N) \le N_0 \Big\} \le C_d (\log N_0)^{-d}. \tag{4}$$

The same exponent d occurs in the underlying first-passage transport bound. In particular one may take

$$d_0 = \frac{6993}{200000} = 0.034965, \qquad \frac{5}{143} - d_0 = \frac{1}{28600000} > 0. \tag{5}$$

The density prefactors implicit in the proof, and the sufficiently-large threshold in the transport theorem, are existential. The exponent d_0, the auxiliary no-hit exponent $1/32000$, and both clocks in (3) are explicit. The exponent $1/32000$ controls a different, polynomial no-hit error; it is not part of the headline logarithmic rate.

In (4), the bad event means exactly that the timed orbit has no value at most N_0, or equivalently that its timed minimum remains strictly above N_0.

There is also a checked lower clock for the raw map.

Corollary 1.3 (Raw-time order optimality)**.** *For a natural-density-one set of positive integers N, there is an $m \in \mathbb{N}$ satisfying*

$$\frac{\log N}{2\log 2} < m \le C_{\mathrm{Coll}} \log N \le 436 \log N, \qquad \mathrm{Col}^m(N) < \sqrt{N}. \tag{6}$$

For every positive N, whether or not it lies in this density-one set, every $m \in \mathbb{N}$ for which $\mathrm{Col}^m(N) < \sqrt{N}$ automatically satisfies the first inequality in (6). Consequently the $\log N$ order of the raw time budget in theorem 1.1 is optimal up to a multiplicative constant.

This is an order-of-growth statement, not a claim that the constant 436 is sharp. For comparison, Inselmann's fixed-power trajectory results use the much smaller horizons $3/\log(4/3) \approx 10.43$ raw steps and $1/\log(4/3) \approx 3.48$ Syracuse steps per $\log N$, versus the present bounds 436 and 145; the tradeoff here is that one clock works for every diverging target, not only the target classes treated there. Nor is there a comparable pointwise lower bound for the Syracuse clock: one Syracuse step can remove an arbitrarily large power of two.

The proof has four conceptual stages.

(1) A natural-counting block law is compared with the logarithmic block law used in Tao's first-passage machinery. A phase-discrepancy estimate preserves every $d < 5/143$.
(2) A single top-block splice converts this transport theorem into the uniform fixed-target estimate (4). A geometric schedule records the time spent at every passage scale.
(3) Freezing a sufficiently large target turns the fixed-target estimate into a density-one theorem for every growing f, with the same Syracuse clock.
(4) A finite two-adic-fiber argument transfers the result to the raw map. An exact valuation telescope supplies the raw clock.

The analytic and arithmetic proof chain has been formalized in Lean 4. The formal development includes the Padé/irrationality-measure input, rather than taking a phase-gap hypothesis as an axiom. It also checks that forgetting the clocks recovers the previously frozen untimed natural-density theorem. The formal trust boundary is described in section 8.

The band-local transport argument is substantially longer than its downstream applications. Section 3 now gives its mathematical proof in a reader-facing sequence: phase discrepancy, common band profile, exact source mixture, totalization, and the real-endpoint adapter. The technical appendices record the longer finite identities, all constants and floor conventions, and the exact correspondence with Lean declarations. The natural-counting splice, clock extraction, growing-threshold argument, two-adic transfer, and raw lower bound are then given explicitly below.

This work does not prove the Collatz conjecture for every input. Natural density one permits an exceptional set of density zero. No optimality is claimed for the exponent or either upper-clock constant.

2. Maps, counting conventions, and first passage

We write $\mathbb{N} = \{0, 1, 2, \ldots\}$ and explicitly impose positivity where it is needed. For a set $A \subseteq \mathbb{N}$ and $X \in \mathbb{N}_{>0}$, put

$$D_X(A) = \frac{1}{X}\#(A \cap [0, X)), \qquad \mathrm{dens}(A) = \lim_{X \to \infty} D_X(A) \tag{7}$$

when the limit exists. Thus the machine-checked natural count is half-open. The fixed-target numerator in (4) is instead the inclusive real-endpoint count through $\lfloor x \rfloor$; its positive odd guard removes zero.

For a property P of odd integers, "relative natural density one among the odds" is encoded by

$$\mathrm{dens}\,\{N \in \mathbb{N} : N \text{ odd and } P(N)\} = \frac{1}{2}. \tag{8}$$

Since the odd integers themselves have natural density 1/2, this is equivalent to the usual odd-relative formulation.

The Syracuse map in (2) is used only on odd inputs in the mathematical statements. In the Lean library it is a total function on $\mathbb{N}$; on even inputs its inherited definition is not the usual odd-to-odd map. Every Syracuse conclusion consumed here carries an odd-input guard, so this totalization does not affect the result.

Fix $x \geq 1$. For an odd N, define the extended-natural first-passage time by

$$T_x(N) = \begin{cases} \min\{m \in \mathbb{N} : \mathrm{Syr}^m(N) \leq x\}, & \{m \in \mathbb{N} : \mathrm{Syr}^m(N) \leq x\} \neq \varnothing, \\ \infty, & \text{otherwise.} \end{cases} \tag{9}$$

The first-passage location is $\mathrm{Pass}_x(N) = \mathrm{Syr}^{T_x(N)}(N)$ on a hit and is totalized to the distinguished value 1 on no hit. This follows Tao's deliberately artificial totalization convention and makes the pass law a probability measure on $[1, \lfloor x \rfloor]$ without silently discarding escape mass.

Following Tao's concrete choice, set

$$\alpha = \frac{1001}{1000}.$$

For $y \geq 1$, let

$$B_y = \{N \in \mathbb{N} : N \text{ odd}, \ \lceil y \rceil \leq N \leq \lfloor y^\alpha \rfloor\}.$$

Whenever $B_y \neq \varnothing$, we use two laws on this same finite block. The natural-counting law is

$$U_y = \mathrm{Unif}(B_y),$$

while the logarithmic law is

$$L_y(N) = \frac{N^{-1}}{\sum_{M \in B_y} M^{-1}} \qquad (N \in B_y).$$

Every use below is in a regime where nonemptiness has been established. Expressions such as $T_x(U_y)$ mean that the displayed function is evaluated at a random input with law U_y; $\mathcal{L}$ denotes the resulting law. The core transport problem is to compare the pushforwards of these laws by Pass_x, while also bounding the uniform probability of missing $[1, x]$ within the scheduled time.

We use the full ℓ^1 distance

$$\|\mu - \nu\|_1 = \sum_z |\mu(z) - \nu(z)|. \tag{10}$$

This is twice the alternative normalization of total variation. Only fixed absolute factors depend on this convention.

3. The quantitative transport bridge

The main analytic input compares natural and logarithmic first-passage laws on the same block. It is useful to state its two error terms separately. For $x \geq 2$, write

$$s(x) = \left\lfloor \frac{\log \lfloor x \rfloor}{10 \log 2} \right\rfloor.$$

Set

$$C_{\mathrm{hit}} = 184, \qquad C_{\mathrm{tr}} = 384256. \tag{11}$$

Theorem 3.1 (Two-block transport and scheduled hitting)**.** *Fix d with $0 < d < 5/143$. There is an $x_0 \geq 2$ such that, whenever $x \geq x_0$, the following bounds hold with the explicit, certified but not optimized constants in* (11) *for each*

$$y \in \left\{x^\alpha, x^{\alpha^2}\right\}.$$

The block B_y is nonempty, and

$$\mathbb{P}(T_x(U_y) > s(x)) \le C_{\mathrm{hit}} x^{-1/32000}, \tag{12}$$

$$\|\mathcal{L}(\mathrm{Pass}_x(U_y)) - \mathcal{L}(\mathrm{Pass}_x(L_y))\|_1 \le C_{\mathrm{tr}}(\log x)^{-d}. \tag{13}$$

The same literal d in (13) is available in theorem 1.2.

The passage clause (13) is the checked ND-M2 transport component. ND-M2's weaker never-hit bound and the separate stronger scheduled bound (12) both admit the witness 184; equal witnesses do not identify the events.

The two clauses play different roles. The scheduled no-hit estimate (12) is a genuine negative power of x. The transport estimate (13) is a negative power of $\log x$, and its exponent is limited by a Diophantine phase estimate. Since

$$x^{-a} = o((\log x)^{-d}) \qquad (a, d > 0),$$

the no-hit exponent $1/32000$ can be absorbed into any fixed logarithmic rate satisfying the hypotheses. It is therefore kept explicit but separate.

3.1. **The Rhin phase input.** For $t \in \mathbb{R}$, let

$$\|t\|_{\mathbb{R}/\mathbb{Z}} = \inf_{k\in\mathbb{Z}} |t - k|\,.$$

The phase estimate consumed by the transport proof has the form

$$\|q \log_2 3\|_{\mathbb{R}/\mathbb{Z}} \ge c q^{1-\kappa} \qquad (q \in \mathbb{N}_{>0}), \tag{14}$$

where $0 < c \le 1$ and $\kappa > 2$.

Proposition 3.2 (Rhin phase gap)**.** *There is a $c \in (0, 1]$ for which*

$$\|q \log_2 3\|_{\mathbb{R}/\mathbb{Z}} \ge c q^{-133/10} \qquad (q \in \mathbb{N}_{>0}). \tag{15}$$

Equivalently, (14) holds with $\kappa = 143/10 = 14.3$.

Proof. Rhin's Proposition [13, p. 160, (7)] states that, for integers u_0, u_1, u_2 with $H = \max(|u_1|, |u_2|) \ge 2$, the form

$$\Lambda = u_0 + u_1 \log 2 + u_2 \log 3$$

satisfies $|\Lambda| \ge H^{-13.3}$. We use only the special case $(u_0, u_1, u_2) = (0, -p, q)$. Given $q \ge 1$, choose a nearest integer p to $q \log_2 3$. The form $q \log 3 - p \log 2$ is nonzero, since no positive power of 2 equals a positive power of 3. Since $3/2 < \log_2 3 < 2$, the nearest-integer choice gives $2 \le H = \max(|p|, q) = \max(p, q) \le 2q$. Hence

$$\|q \log_2 3\|_{\mathbb{R}/\mathbb{Z}} = \frac{|q \log 3 - p \log 2|}{\log 2} \ge \frac{2^{-13.3}}{\log 2}\, q^{-13.3}.$$

Capping this constant by 1 proves the claim. Thus the source is used only through a legitimate weakening of its constant-one three-term statement. □

The same 13.3 proposition has previously been used in the Collatz literature to bound hypothetical cycles [14]. Its role here is different: it controls the discrepancy of the irrational rotation that appears when natural and reciprocal source weights are compared. For formal verification we do not install Rhin's printed proposition as an axiom. Appendix E explains how the required weaker large-height theorem is reconstructed from his Padé construction, after which the finite range is absorbed symbolically by nonvanishing. No historical low-frequency phase table is used.

3.2. **From the phase gap to endpoint discrepancy.** Put $\theta = \log_2 3$. For an anchor $\phi \in \mathbb{R}/\mathbb{Z}$, consider the length-$N$ orbit $\phi + n\theta$, $0 \le n < N$, and write $u_n \in [0,1)$ for its canonical representative. For $N \ge 1$ and $\diamond \in \{<, \le\}$, define

$$D_N^{\diamond}(\phi) = \sup_{0\le t\le 1} \left| \frac{1}{N} \#\{n < N : u_n \diamond t\} - t \right|, \qquad D_N(\phi) = \max\{D_N^{<}(\phi), D_N^{\le}(\phi)\}. \tag{16}$$

Keeping both endpoint conventions is useful because the passage bands are half-open while several finite target windows are inclusive.

Lemma 3.3 (Phase discrepancy)**.** *If* (14) *holds, then, for every* $N \ge 1$ *and every anchor* ϕ*,*

$$D_N(\phi) \le \min\left\{1, \left(6 + \frac{2}{\pi c}\right) N^{-1/\kappa}\right\}. \tag{17}$$

Proof. For $q \ge 1$, the finite geometric sum and the nearest-integer chord bound give

$$\left| \frac{1}{N} \sum_{n<N} e^{2\pi i q(\phi + n\theta)} \right| \le \min\left\{1, \frac{1}{2Ncq^{1-\kappa}}\right\}. \tag{18}$$

The anchored Erdős–Turán inequality, in the normalization used here, is

$$D_N(\phi) \le \frac{6}{H+1} + \frac{4}{\pi} \sum_{q=1}^{H} \frac{1}{q} \left| \frac{1}{N} \sum_{n<N} e^{2\pi i q(\phi+n\theta)} \right|.$$

Using the second term of (18) and $\sum_{q\le H} q^{\kappa-2} \le H^{\kappa-1}$, we obtain

$$D_N(\phi) \le \frac{6}{H+1} + \frac{2}{\pi N c} H^{\kappa-1}.$$

Choose $H = \left\lceil N^{1/\kappa} \right\rceil - 1$ when $N > 1$, and use the trivial unit bound when $N = 1$. Both displayed terms are then bounded by their corresponding coefficients times $N^{-1/\kappa}$, proving (17). □

This is the classical Erdős–Turán discrepancy mechanism; see, for example, [9, Chapter 2]. The strict/weak endpoint version and the displayed constants used here are proved directly, both in the argument above and in Lean.

This elementary optimization is the only place where κ first enters. The later passage geometry has a tube of length comparable with the square root of the scheduling scale, so the exponent is halved once more.

3.3. **Exact bands and the common passage profile.** We now give the transport proof. It is convenient to work first at a natural target B and then pass to a real target. There are two source branches

$$y_b \in \{B^{\alpha}, B^{\alpha^2}\}.$$

For one branch put

$$J_b = \lfloor (\alpha - 1)\log y_b \rfloor, \qquad \beta_b = \frac{(\alpha-1)\log y_b}{J_b}, \qquad z_{b,j} = y_b e^{j\beta_b}.$$

For sufficiently large B, $J_b > 0$, and the odd source block is the disjoint union of the half-open bands

$$\mathcal{B}_{b,j} = \{N \text{ odd} : z_{b,j} \le N < z_{b,j+1}\}, \qquad 0 \le j < J_b, \tag{19}$$

and a possible odd integer lying exactly at the final real endpoint. On each $\mathcal{B}_{b,j}$ let $\mu_{b,j}^{\mathrm{F}}$ be the uniform law and $\mu_{b,j}^{\mathrm{H}}$ the law proportional to $1/N$. The superscripts stand for flat and harmonic. After enlarging the common lower threshold for B, every band with $0 \le j < J_b$ is nonempty.

The common profile can be written explicitly. Let

$$n_0(B) = \left\lfloor \frac{\log_2 B}{10} \right\rfloor, \qquad m_0(B) = \left\lfloor \frac{\log B}{100000} \right\rfloor, \qquad m_B = \left\lceil m_0(B)(\log B)^{-2/5} \right\rceil.$$

These parameters have different jobs. The quantity $n_0(B)$ is the local Section 5 passage horizon and determines the valuation-tube width; $m_0(B)$ is the exact step-back time used to define the lost-window target; and m_B is the shorter affine reference depth in the common kernel Z_{m_B}. In particular, the three symbols are not alternate names for one schedule. The scale is deliberately asymptotic: $m_0(B) = 0$ for $1 \le B < e^{100000}$, so this construction does not supply a practical finite threshold. To define the affine law precisely, let $a_1, \dots, a_m$ be independent positive geometric variables with $\mathbb{P}(a_i = k) = 2^{-k}$. For a tuple $a = (a_1, \dots, a_m)$, put $w(a) = \sum_i a_i$ and define

$$b(\varnothing) = 0, \qquad b(a_1, \dots, a_m) = 3^{m-1} + 2^{a_1} b(a_2, \dots, a_m).$$

In $\mathbb{Z}/3^m\mathbb{Z}$, set $A_m(a) = b(a)2^{-w(a)}$, and let $p_m(r) = \mathbb{P}(A_m(a) = r)$. Thus p_m is the exact recursive Syracuse affine law, not an asymptotic model. Define

$$Z_m(F) = \sum_{M \in F} \frac{3^m p_m(M \bmod 3^m)}{M}. \tag{20}$$

Put

$$g_B = (\log B)^{7/10}, \qquad \Lambda_B = (4/3)^{m_0(B)} B,$$

and let the canonical lost window be $\mathcal{W}_B = \mathbb{N} \cap [e^{-g_B}\Lambda_B, e^{g_B}\Lambda_B]$. For an output event $E \subseteq \{1, \dots, B\}$, define the common finite step-back target exactly by

$$\begin{aligned} E'_B(E) = \{M \in \mathcal{W}_B : M \text{ odd},\ \mathrm{Syr}^{m_0(B)}(M) \le B,& \\ \mathrm{Syr}^k(M) > B\ (0 \le k < m_0(B)),\quad \mathrm{Syr}^{m_0(B)}(M) \in E\}&. \end{aligned} \tag{21}$$

Thus the first hit occurs at exactly $m_0(B)$, not merely sometime before that horizon. Set

$$\mathcal{Z}_B(E) = \frac{2}{\log(4/3)} Z_{m_B}(E'_B(E)). \tag{22}$$

The key point is visible in the notation: $\mathcal{Z}_B(E)$ depends on neither the source branch, the band, nor the choice of flat or harmonic law. For all sufficiently large B,

$$0 \le \mathcal{Z}_B(E) \le 40. \tag{23}$$

Indeed, the payload-free common kernel in (20) is at most 5, while $2/\log(4/3) \le 8$. This is checked by `Tao.taoSection5PayloadFreeCommonZ_ePrime_mem_Icc_five` and `eventually_ndA5ReferenceCommonProfile_mem_Icc_forty`. The resulting profile cap $40 = 8 \cdot 5$ is unrelated to the later conservative tube choice $C = 40$.

Write $P^\sigma_{b,j}(E)$ for the genuine first-passage mass of E from $\mu^\sigma_{b,j}$, with no artificial mass assigned on escape, and put

$$R_d(B) = (\log B)^{-d}.$$

Let $I^\sigma_{b,j}(E)$ denote the scheduled reference profile restricted to the moderate interior endpoint set, and let $F^\sigma_{b,j}(E)$ denote the corresponding full physical reference profile, interior plus exterior, on the same exact carrier. These are the two profiles whose literal finite sums are expanded in appendix A.

Lemma 3.4 (Interior and full reference profiles)**.** *Assume* (14)*, $d > 0$, and $d < 1/(2\kappa)$. For all sufficiently large B, uniformly in b, j, E, σ,*

$$\left| I^\sigma_{b,j}(E) - \mathcal{Z}_B(E) \right| \le 5R_d(B), \tag{24}$$

$$\left| F^\sigma_{b,j}(E) - \mathcal{Z}_B(E) \right| \le 6R_d(B). \tag{25}$$

Proof. We separate the analytic estimate from the passage bookkeeping. Fix a tube constant $C \ge 1/2$, later $C = 40$, and write

$$W = C\sqrt{n_0(B) \log n_0(B)}, \qquad \rho(B) = \frac{(\log n_0(B))^{3/2}}{\sqrt{n_0(B)}}.$$

The moderate physical window has logarithmic radius $(4/25)W = O(\sqrt{\log B \log\log B})$. It is therefore eventually contained in the canonical lost window, whose radius is $(\log B)^{7/10}$. The interior/exterior split takes place inside the gap between these scales. The affine endpoint calculation is performed on the physical tube of valuation levels associated with a fixed terminal endpoint M. The tube has cardinality between W and $7W$. Applying lemma 3.3, finite Abel summation, and the local Gaussian estimate for the Syracuse affine model gives, for both normalized profiles, the endpoint-independent majorant

$$\begin{aligned}\mathcal{E}_{B,C} = {}& \frac{39927552}{25}\left(6+\frac{2}{\pi c}\right)C^2(\log n_0(B))W^{-1/\kappa} \\ &+\left(\frac{101856}{25}C+288\right)\rho(B) \\ &+54750000(C+C^3)\rho(B)+\frac{12}{B\log(4/3)}.\end{aligned} \tag{26}$$

The first line is the phase-discrepancy contribution. Since $n_0(B) \asymp \log B$ and $W = C\sqrt{n_0(B)\log n_0(B)}$, it has size

$$O_{c,\kappa,C}\left((\log B)^{-1/(2\kappa)}(\log\log B)^{1-1/(2\kappa)}\right).$$

Thus it is $o(R_d(B))$ precisely when $d < 1/(2\kappa)$. The $\rho(B)$ and B^{-1} terms decay faster.

We next record how this scalar estimate becomes a common profile. Write R_H, R_F for the unnormalized reciprocal and flat two-profile sums on the same physical carrier. Appendix A defines their summands and produces explicit nonnegative bounds $E_H^{\mathrm{raw}}, E_F^{\mathrm{raw}}$ satisfying

$$|R_H - \beta_b/\log(4/3)| \le E_H^{\mathrm{raw}}, \qquad |R_F - (e^{\beta_b}-1)/\log(4/3)| \le E_F^{\mathrm{raw}}.$$

The exact band normalizers then place both normalized physical profiles at the same center $2/\log(4/3)$. For a band with lower endpoint z, their finite estimates are

$$\left|H_{\mathrm{norm}} - \frac{2}{\log(4/3)}\right| \le 4E_H^{\mathrm{raw}} + \frac{12}{z\log(4/3)},$$
$$\left|F_{\mathrm{norm}} - \frac{2}{\log(4/3)}\right| \le 3E_F^{\mathrm{raw}} + \frac{6}{z\log(4/3)}.$$

Both right sides are bounded by the single quantity $\mathcal{E}_{B,C}$ in (26); the appendix gives the literal formulas and the exact finite inequality, not only their asymptotic orders. Multiplication by the common kernel in (20) and summation over the original endpoint set introduce no cardinality loss. The interior endpoints contribute at most $5R_d(B)$. The exterior profile, together with the omitted exterior copy of its center, contributes one more $R_d(B)$. This proves (24) and (25). □

Lemma 3.5 (Terminal reconstruction)**.** *Assume the hypotheses of lemma 3.4 and also $d < 1/20$. At the fixed tube constant $C = 40$, one eventual threshold works uniformly in b, j, E, σ, and*

$$\left|P^{\sigma}_{b,j}(E) - F^{\sigma}_{b,j}(E)\right| \le \left(\frac{1}{4}+\frac{1}{2}+\frac{1}{4}\right)R_d(B) = R_d(B). \tag{27}$$

Proof. The terminal-approximation errors have decay powers

$$\frac{1}{20},\ \frac{3}{10},\ \frac{9}{5}, \qquad 6,\ \frac{6}{5};$$

so the first is the binding strict guard. Comparing the actual first-passage event with the full physical reference profile has exactly three charged terms. The failure of terminal coverage costs $R_d(B)/4$, replacement by nominal terminal masses costs $R_d(B)/2$, and aggregation at the common reference level costs $R_d(B)/4$. Their sum is (27). More precisely, the harmonic coverage error is at most $\frac{1}{4}(\log B)^{-10}$, and the flat error is strictly below $(\log B)^{-10}$; both are eventually at most $R_d(B)/4$. Exact-to-nominal replacement is bounded by $11B^{-9/10} \le R_d(B)/2$, and the two nominal sums differ from their physical profiles by at most $R_d(B)/4$.

These three inputs are the coverage lemmas in `A5BandCoverageRate.lean`, the terminal-atom lemmas in `A5ReferenceGenuineBandRate.lean`, and `eventually_abs_ndA5ReferenceTerminalNominalSums_sub_physicalProfiles_le_rate`, respectively. □

The following proposition is the central local statement.

Proposition 3.6 (Common band profile). *Assume* (14)*, and suppose*

$$0 < d < \frac{1}{20}, \qquad d < \frac{1}{2\kappa}. \tag{28}$$

Then, for all sufficiently large B, uniformly in the two branches, every band $0 \le j < J_b$, every output event E, and $\sigma \in \{\mathrm{F}, \mathrm{H}\}$,

$$\left| P^{\sigma}_{b,j}(E) - \mathcal{Z}_B(E) \right| \le 7R_d(B). \tag{29}$$

Proof. Apply lemmas 3.4 and 3.5 and the triangle inequality:

$$|P^{\sigma}_{b,j}(E) - \mathcal{Z}_B(E)| \le |P^{\sigma}_{b,j}(E) - F^{\sigma}_{b,j}(E)| + |F^{\sigma}_{b,j}(E) - \mathcal{Z}_B(E)| \le R_d(B) + 6R_d(B).$$

□

The theorem map in the appendices records the load-bearing exported endpoints of this ledger; the supporting declarations are identified where their estimates are used.

The proof displays why a phase estimate is needed for natural counting. Reciprocal weighting already matches the logarithmic architecture of the first-passage theorem. Flat weighting changes the distribution of endpoints inside every multiplicative band. The irrational rotation generated by $\log_2 3$ makes those endpoint phases equidistributed with the quantitative error (17); the common center then removes the law, band, and branch labels simultaneously.

3.4. Whole-source mixture and full ℓ^1 distance.

Let $P^{\sigma}_b(E)$ be the passage mass from the entire source block. The half-open partition (19) gives the exact event identity

$$P^{\sigma}_b(E) = \sum_{j<J_b} w^{\sigma}_{b,j} P^{\sigma}_{b,j}(E) + T^{\sigma}_b(E), \tag{30}$$

where

$$\sum_{j<J_b} w^{\sigma}_{b,j} + t^{\sigma}_b = 1, \qquad 0 \le T^{\sigma}_b(E) \le t^{\sigma}_b, \qquad t^{\sigma}_b \le \frac{4}{B}. \tag{31}$$

The last term is the possible odd top endpoint. It is deliberately kept as a submass rather than normalized into a probability law, since its carrier may be empty.

By (29) and the exact normalization in (31), the band sum differs from $(1 - t^{\sigma}_b)\mathcal{Z}_B(E)$ by at most $7R_d(B)$. From (23),

$$|T^{\sigma}_b(E) - t^{\sigma}_b \mathcal{Z}_B(E)| \le 40t^{\sigma}_b \le \frac{160}{B} \le R_d(B)$$

eventually. We have therefore proved

$$|P^{\sigma}_b(E) - \mathcal{Z}_B(E)| \le 8R_d(B). \tag{32}$$

There is no factor equal to the number of bands: the weights, rather than unweighted errors, are what sum to one.

The triangle inequality now gives $16R_d(B)$ both for flat versus harmonic passage on one branch and for harmonic passage on the two different branches. To include escape mass, put every passage law on the common finite carrier $\{M \in \mathbb{N} : M \le B\}$, assigning the distinguished value 1 on no hit. For events not containing 1, event mass is exactly the genuine passage mass. For events containing 1, take the complement inside this finite carrier. One finite Hahn decomposition then converts the uniform event bound into

$$\|\mathcal{L}(\mathrm{Pass}_B(U)) - \mathcal{L}(\mathrm{Pass}_B(L))\|_1 \le 32R_d(B). \tag{33}$$

This is full ℓ^1, hence the single factor two.

The scheduled no-hit clause. This part is independent of the phase argument. On each exact band, flat density is at most e^{β_b} times harmonic density, while $\beta_b < 2\log 2$, so $e^{\beta_b} \le 4$. Tao's scheduled reference error at $n_0(B)$ is at most $10B^{-1/32000}$; hence each flat band contributes at most $40B^{-1/32000}$. Exact convex mixing creates no band-count factor, and the possible top endpoint contributes at most $4/B \le 4B^{-1/32000}$. Thus the natural-target source bound is $44B^{-1/32000}$. The real-floor comparison in (35) below turns this into the stated constant 184. The theorem map records the three checked scheduled declarations, including the real-source adapter to (184); the underlying schedule is Tao's [15, (5.1)–(5.2)].

3.5. **Real endpoints and completion of transport.** It remains to replace the natural threshold and source endpoints by the real ones in theorem 3.1. Put $B = \lfloor x \rfloor$ and

$$\delta_U = 96x^{-99/100}, \qquad \delta_H = \frac{96000}{B\log B}. \tag{34}$$

The exact uniform-source and harmonic-source floor comparisons give the raw ledger

$$\mathbb{P}(T_x(U_y) > s(x)) \le 44B^{-1/32000} + \delta_U, \tag{35}$$

$$\|\mathcal{L}(\mathrm{Pass}_x(U_y)) - \mathcal{L}(\mathrm{Pass}_x(L_y))\|_1 \le 2\delta_U + 32R_d(B) + 2\delta_H. \tag{36}$$

The second line is exactly the triangle

$$U_{\mathrm{real}} \longrightarrow U_{\mathrm{floor}} \longrightarrow H_{\mathrm{floor}} \longrightarrow H_{\mathrm{real}}.$$

For all sufficiently large x, $x/2 \le B$ and $(\log x)/2 \le \log B$. Monotonicity of negative powers, together with the elementary fact that every fixed power of x^{-1} eventually beats every fixed power of $(\log x)^{-1}$, turns (35) into

$$44B^{-1/32000} + 96x^{-99/100} \le 184x^{-1/32000},$$

and (36) into

$$\begin{aligned} 2(96x^{-99/100}) + 32(\log B)^{-d} + \frac{192000}{B\log B} &\le (192 + 64 + 384000)(\log x)^{-d} \\ &= 384256(\log x)^{-d}. \end{aligned}$$

Thus the constants stated in theorem 3.1 are exactly

$$C_{\mathrm{hit}} = 184, \qquad C_{\mathrm{tr}} = 384256, \tag{37}$$

after enlarging one threshold x_0. These displayed transport prefactors are explicit and checked in the formal theorem; later density prefactors also include geometric series and finite-range absorptions and are left existential.

3.6. **Where the exponent enters.** The band-local analysis is parametric in (c, κ). For every fixed $\varepsilon > 0$, its phase line has the limiting form

$$(\log x)^{-1/(2\kappa)+\varepsilon}. \tag{38}$$

All other governing errors allow the additional harmless restriction $d < 1/20$. Thus the formal transport theorem accepts

$$0 < d < \frac{1}{20}, \qquad d < \frac{1}{2\kappa}. \tag{39}$$

For $\kappa = 143/10$,

$$\frac{1}{2\kappa} = \frac{5}{143} = 0.0349650349\ldots < \frac{1}{20},$$

so (39) reduces to $0 < d < 5/143$. This is why the exact margin in (5) matters: 0.034965 is below the open cap, not an alternate notation for it.

We summarize the analytic proof of theorem 3.1. On each source block, the Syracuse valuation word is grouped first by its fiber data and then by phase. Lemma 3.3 controls the phase orbit; the affine and Gaussian estimates produce (26); the common-center calculation and terminal reconstruction give (29). Exact source mixing, top-endpoint control, finite totalization, and the real-floor triangle then give (37). The scheduled band argument above gives (12); the never-hit event is a subset of this scheduled event. Keeping the power error separate prevents the no-hit exponent from degrading the caller-selected d.

For orientation, the load-bearing chain is

$$\begin{aligned}&\text{Rhin phase gap} \Longrightarrow \text{phase average} \Longrightarrow \text{normalized band profile}\\ &\Longrightarrow \text{exact source mixture and totalization} \Longrightarrow \text{two-block passage transport.}\end{aligned} \tag{40}$$

The appendices expand the finite affine estimate, the interior/exterior split, and every constant in the $7 \to 8 \to 16 \to 32$ ledger. No numerical experiment is promoted to a theorem at any arrow of (40).

4. From transport to a timed fixed-target estimate

We now prove the natural-counting statement theorem 1.2. The important point is that the proof records time while carrying the same transport exponent.

4.1. The scheduled clock. At a natural threshold $B \geq 1$, the passage machinery uses the same integer schedule as the common profile,

$$n_0(B) = \left\lfloor \frac{\log_2 B}{10} \right\rfloor = \left\lfloor \frac{\log B}{10 \log 2} \right\rfloor. \tag{41}$$

On both transport branches, failure to hit $\lfloor x \rfloor$ by this schedule has a uniform-block bound of the form

$$\mathbb{P}(T_x(U_y) > n_0(\lfloor x \rfloor)) \leq 184 x^{-1/32000} \tag{42}$$

for all sufficiently large x. Here 184 is the explicit checked constant C_{hit} from (37), not a coincidental reuse; retaining the power exponent is useful.

The time-carrying form of the Section 3 telescope is the following lemma.

Lemma 4.1 (Timed geometric trace)**.** *Suppose* (14) *holds for* c, κ, *and let* d *satisfy the two strict guards in* (28). *There are* $C_{\text{trace}} \geq 0$ *and* $N_* \geq 2$ *such that, for* $N_0 \geq N_*$ *and* $X > N_0$, *some* $T \in \mathbb{N}$ *satisfies*

$$T \leq \frac{1001}{10 \log 2} \log X, \tag{43}$$

$$\mathbb{P}_{N \sim L_{X^r}}(\nexists m \leq T : \mathrm{Syr}^m(N) \leq N_0) \leq C_{\text{trace}} (\log N_0)^{-d}, \qquad r = \alpha^{-1}. \tag{44}$$

Proof. Put $b_i = X^{r^i}$, and let K be the first index for which $b_K \leq N_0^r$. The least-crossing inequalities force $K = j + 2$ for a unique $j \geq 0$, with

$$b_{j+1} \leq N_0, \qquad b_{j+3} > N_0^{r^3}.$$

Set

$$y_* = b_{j+2}, \qquad J = j + 1, \qquad q_\ell = (y_*^{\alpha^\ell})^r \quad (0 \leq \ell \leq J),$$

and define the accumulated integer clock by

$$T = \sum_{\ell=0}^{J-1} n_0(\lfloor q_\ell \rfloor). \tag{45}$$

The identities $q_{\ell+1} = q_\ell^\alpha$, $q_0 = b_{j+3}$, and $y_*^{\alpha^J} = X^r$, equivalently $q_J = X^{r^2}$, show respectively that the active logarithms form the required geometric progression, that $\log q_0 > r^3 \log N_0$, and that the terminal source window is exactly the block carrying L_{X^r}.

At step ℓ, one constant C_{step}, independent of ℓ, N_0, X, bounds the combined scheduled-escape and passage-transport error by $C_{\text{step}} (\log q_\ell)^{-d}$. Therefore

$$C_{\text{step}} (r^3)^{-d} \sum_{\ell \geq 0} \alpha^{-\ell d} (\log N_0)^{-d} = \frac{C_{\text{step}} (r^3)^{-d}}{1 - \alpha^{-d}} (\log N_0)^{-d}.$$

This is (44). By (41), each summand in (45) is at most $(10\log 2)^{-1}\log q_\ell$. The finite geometric identity and $\alpha - 1 = 1/1000$ give $\sum_{\ell<J}\log q_\ell \le 1001\log q_J \le 1001\log X$, proving (43). A single lower threshold absorbs the finite prefix and supplies nonempty logarithmic source blocks. The least crossing, definitions of q_ℓ, J, T, probability telescope, and clock inequality are checked together in `exists_ndA5ReferenceSection3AmbientTimeTerminal`. □

Put $r = \alpha^{-1} = 1000/1001$ and $Q = X^{r^2}$. At the terminal top block, one final passage schedule costs at most $(10\log 2)^{-1}\log X$. Hence the ambient cost is

$$n_0(\lfloor Q\rfloor) + T \le \frac{1002}{10\log 2}\log X. \tag{46}$$

Every integer in the top block $[X^r, X]$ satisfies $r\log X \le \log N$, hence the exact conversion $\log X \le \alpha\log N$. Applying it to (46) gives

$$n_0(\lfloor Q\rfloor) + T \le \frac{1001}{1000}\frac{1002}{10\log 2}\log N = \frac{501501}{5000\log 2}\log N = C_{\mathrm{Syr}}\log N.$$

This proves that the same pointwise clock applies throughout the counted top block.

4.2. **The top-block splice.** Fix a target N_0 and an endpoint $X > N_0$. Put

$$Q = X^{r^2}, \qquad y = Q^\alpha = X^r;$$

then B_y is the odd top block ending at X. The lower interval $[1, \lceil y\rceil)$ is discarded. On the top block, the timed bad probability is controlled by exactly three contributions:

(a) failure of the uniform source to reach Q by the schedule $n_0(\lfloor Q\rfloor)$;
(b) the error from replacing the uniform first-passage law by the logarithmic first-passage law;
(c) failure of a logarithmically weighted source input to reach N_0 during the remaining trace from Tao's Section 3.

Choose $T = T(N_0, X) \in \mathbb{N}$ from lemma 4.1, and define

$$\begin{aligned} E_{\mathrm{sched}}(Q) &= \mathbb{P}_{N\sim U_y}(\nexists m \le n_0(\lfloor Q\rfloor) : \mathrm{Syr}^m(N) \le Q), \\ E_{\mathrm{pass}}(Q, y) &= \|\mathcal{L}(\mathrm{Pass}_Q(U_y)) - \mathcal{L}(\mathrm{Pass}_Q(L_y))\|_1, \\ E_{\mathrm{target}}(N_0, X) &= \mathbb{P}_{N\sim L_y}(\nexists m \le T : \mathrm{Syr}^m(N) \le N_0). \end{aligned}$$

Failure to hit N_0 within $n_0(\lfloor Q\rfloor) + T$ is contained in scheduled failure, or a bad passage endpoint. After transporting the endpoint law, the probability of a bad logarithmic passage endpoint is bounded by E_{target}: its preimage under the pass map is contained in the original logarithmic-source T-step bad event. This is the exact bridge proved by `pmfProb_fixedTargetBadWithin_add_le` in `ND/LogTime/FixedTargetBridge.lean`.

The pass discrepancy is already measured in the full-ℓ^1 convention (10). Separately, the checked rounding estimate

$$\#B_y \le \lfloor X\rfloor + 1 \le X + 1 \le 2X$$

converts uniform block probability to an ambient-endpoint count and supplies the outer factor 2. This is the actual provenance of that factor in `oddSyracuseLogTimeBadTopRatio_le_two_mul_uniformWithinBad`, not a heuristic estimate of the number of odd integers. The formal count adapter yields

$$\begin{aligned} \frac{\#\{\text{timed bad odd } N \le \lfloor X\rfloor\}}{X} &\le \frac{\lceil y\rceil}{X} \\ &\quad + 2\big(E_{\mathrm{sched}}(Q) + E_{\mathrm{pass}}(Q, y) + E_{\mathrm{target}}(N_0, X)\big). \end{aligned} \tag{47}$$

The inequality direction is important: natural-counting bad mass is bounded by logarithmic-source bad mass plus discarded lower mass, never conversely.

Including the discarded lower interval, the four terms in (47) satisfy

$$\frac{\lceil y\rceil}{X} \ll X^{-1/32000}, \tag{48}$$

$$E_{\mathrm{sched}}(Q) \ll Q^{-1/32000}, \tag{49}$$

$$E_{\mathrm{pass}}(Q,y) \ll (\log Q)^{-d}, \tag{50}$$

$$E_{\mathrm{target}}(N_0,X) \ll (\log N_0)^{-d}. \tag{51}$$

Here the implicit constants depend only on the fixed exponent d and the fixed Rhin phase-gap witness, never on N_0 or X. The last bound is the geometrically summed fixed-target trace. Its series factor is

$$\sum_{j\geq 1}\alpha^{-jd} = \frac{\alpha^{-d}}{1-\alpha^{-d}} < \infty.$$

It can be large for the displayed d_0, but it is a fixed contribution to the existential prefactor, not a change in exponent.

Every negative power in (48)–(49) is eventually bounded by the appropriate logarithmic power. Since $Q = X^{r^2}$ and $N_0 < X$, the remaining logarithmic terms are bounded by a constant multiple of $(\log N_0)^{-d}$. Substitution in (47) proves the main regime

$$\frac{\#\{\text{timed bad odd } N \leq \lfloor X\rfloor\}}{X} \leq C_{\mathrm{main}}(\log N_0)^{-d} \tag{52}$$

for all sufficiently large N_0 and $X > N_0$.

Two elementary branches make the estimate uniform as stated. If $X \leq N_0$, the bad count is zero: the zeroth iterate already satisfies $N \leq N_0$. For the finitely many targets below the main-regime threshold, the ratio is bounded by 2, and one enlarges a single constant by

$$2(\log N_*)^d.$$

This proves theorem 1.2 for every $N_0, x \geq 2$, without a “sufficiently large” escape clause in its statement.

5. Growing thresholds and one uniform Syracuse clock

We next derive part (i) of theorem 1.1. Fix once and for all any d with $0 < d < 5/143$, for example the value d_0 in (5). The argument is short precisely because theorem 1.2 is uniform in the counting endpoint.

Let f tend to infinity along the odds, and let F be the odd inputs having no strict hit below $f(N)$ within the Syracuse clock. Fix $\eta > 0$. Choose an integer $K \geq 2$ so large that

$$C_d(\log K)^{-d} < \frac{\eta}{2}. \tag{53}$$

Growth along the odds supplies B such that

$$K < f(N) \qquad (N \geq B,\ N \text{ odd}).$$

For such N, failure to hit strictly below $f(N)$ implies failure to hit at most K. Therefore

$$D_X(F) \leq \frac{B}{X} + \frac{1}{X}\#\left\{N\in\mathbb{N} : N < X,\ N \text{ odd, and } N \text{ is timed-bad for } K\right\}. \tag{54}$$

The second set is contained in the inclusive real-endpoint count in the fixed-target theorem at endpoint X. By theorem 1.2, its normalized size is at most the left side of (53); for sufficiently large X, the first is also below $\eta/2$. Thus $D_X(F) \to 0$. Removing this zero-density failure set from the odd integers leaves ambient density $1/2$, which is the odd-relative density-one conclusion (8).

The choice of K depends on the density tolerance and on f, but the clock does not: it remains the one constant C_{Syr} defined before f is quantified. This proves the advertised $\exists C\,\forall f$ strength.

6. Two-adic fibers and the raw Collatz clock

Every positive integer has a unique representation

$$N = 2^a M, \qquad a \geq 0, \quad M \text{ odd}.$$

The fiber with fixed a has natural density 2^{-a-1}. For fixed a, the function $M \mapsto f(2^a M)$ still tends to infinity along odd M, so the Syracuse theorem applies on that fiber.

It remains to control raw time uniformly in a. Suppose an odd source M makes k Syracuse steps, and let

$$W = \sum_{j=0}^{k-1} \nu_2(3\,\mathrm{Syr}^j(M) + 1)$$

be the total number of stripped powers of two. The exact raw simulation time from $2^a M$ is

$$j = a + k + W. \tag{55}$$

At that time the simulated values agree exactly:

$$\mathrm{Col}^{a+k+W}(2^a M) = \mathrm{Syr}^k(M). \tag{56}$$

At each odd step,

$$2^{\nu_2(3n+1)}\,\mathrm{Syr}(n) = 3n + 1 \leq 4n.$$

Telescoping gives

$$2^W\,\mathrm{Syr}^k(M) \leq 4^k M, \qquad W \leq 2k + \frac{\log M}{\log 2}. \tag{57}$$

The factorization $N = 2^a M$ gives the exact identity

$$a + \frac{\log M}{\log 2} = \frac{\log N}{\log 2}. \tag{58}$$

If $k \leq C_{\mathrm{Syr}} \log M$, then $\log M \leq \log N$. Combining (55), (57), and (58),

$$j \leq a + 3k + \frac{\log M}{\log 2} \tag{59}$$

$$\leq 3C_{\mathrm{Syr}} \log N + \frac{\log N}{\log 2}. \tag{60}$$

Thus

$$C_{\mathrm{Coll}} = 3C_{\mathrm{Syr}} + \frac{1}{\log 2} = \frac{1509503}{5000 \log 2}. \tag{61}$$

For each fixed a, the good part of the fiber has its full density 2^{-a-1}. Summing the first $A+1$ disjoint fibers gives lower density at least

$$\sum_{a=0}^{A} 2^{-a-1} = 1 - 2^{-A-1}.$$

Letting $A \to \infty$ proves natural density one and completes theorem 1.1.

The closed forms in (3) are now immediate. The elementary checked estimate $0.693 < \log 2$ gives

$$C_{\mathrm{Syr}} < 145, \qquad C_{\mathrm{Coll}} < 436.$$

Numerically the exact coefficients are approximately 144.7026 and 435.5505, respectively. If the budget is written with $\log_2 N$ instead of $\log N$, the exact coefficients become

$$\frac{501501}{5000} = 100.3002, \qquad \frac{1509503}{5000} = 301.9006.$$

7. The lower raw clock

The matching lower order of growth uses only a deterministic inequality.

Lemma 7.1. *For every $n, m \in \mathbb{N}$,*

$$n \le 2^m \operatorname{Col}^m(n). \tag{62}$$

Proof. One raw step never shrinks by more than a factor of two: $n \le 2\operatorname{Col}(n)$. This is equality on the even branch; on the odd branch it follows from $n \le 2(3n+1)$. Iterating proves (62). ☐

Proposition 7.2 (Pointwise raw-time lower bound)**.** *Let $N > 0$ and $m \in \mathbb{N}$, and suppose $\operatorname{Col}^m(N) < b$ for a real b. Then $b > 0$ and*

$$\frac{\log N - \log b}{\log 2} < m. \tag{63}$$

Proof. By lemma 7.1, $N \le 2^m \operatorname{Col}^m(N) < 2^m b$. Positivity of the left side forces $b > 0$. Taking logarithms and dividing by $\log 2 > 0$ gives (63). ☐

Apply theorem 1.1 to $f(N) = \sqrt{N}$. On its density-one good set, the theorem supplies a raw witness below $\sqrt{N}$ with the upper clock. Since $\log \sqrt{N} = \frac{1}{2}\log N$, proposition 7.2 supplies the strict lower inequality for every raw witness below $\sqrt{N}$, and hence for the one supplied by the theorem. This proves corollary 1.3.

The argument is specific to the raw clock. Equality in (62) along a chain of halvings shows why a logarithmic lower budget is unavoidable for sufficiently strong descent. In contrast, a single Syracuse step may divide by 2^v with unbounded v, so no analogous pointwise lower constant follows for Syracuse time.

8. Formalization and trust boundary

The paper-facing import is

```
import Erdos1135.ND.LogTime.Paper
```

in the Lean 4 project [10, 11]. The parametric all-d rate and fixed-target declarations, their exact rational specialization, the bundled paper theorem, and the raw-clock bracket expose all claims made in the abstract. The timed bundle also implies the frozen untimed theorem by set inclusion and density squeeze; the untimed result is not re-assumed.

The reused precursor formalization culminates in

`Erdos1135.Tao.taoAlmostBoundedColMin_checked`,

the strict orbit-minimum form of Tao's published almost-bounded theorem. Its checked development follows the main published spine through the Syracuse valuation estimates, first-passage transport, Tao's Section 3 scale telescope, and Syracuse-to-standard-Collatz transfer. The ND transport, Rhin discharge, and time layers described in this article are additions to that formalized foundation.

The theorem-bearing source is frozen at Git commit

`f386357d453ac4dcf91242b76252d88a5a729906`.

The complete formalization and replay materials are available from

https://proofatlas.ai/sources/natural-density-log-time-collatz/.

It uses Lean `v4.30.0-rc2` and Mathlib commit `5450b53e5ddc75d46418fabb605edbf36bd0beb6`. At this exact frozen commit, the full paper closure was cold-built in a fresh clone and then replayed by `leanchecker -verbose`. The cited theorem declarations use only `propext`, `Classical.choice`, and `Quot.sound`; there is no `native_decide` trust point, placeholder, or project Collatz-conjecture axiom in their dependency cone.

The appendices give the declaration map, certificate provenance, and replay commands; the commit identifies the complete source tree.

Code and data availability. The frozen Lean source, generated exact Rhin certificate, source ledger, and replay record are available at the ProofAtlas formalization page. The version-1 PDF is available at its ProofAtlas paper page. No external empirical data set is used.

9. Context, limitations, and possible improvements

Theorem 1.1 remains an almost-everywhere result: its exceptional set may be infinite, and it does not prove that every Collatz orbit reaches 1. The Syracuse first-passage architecture remains Tao's. The Terras–Everett–Allouche–Korec line already supplies natural-density descent, and Inselmann supplies much smaller logarithmic horizons for every fixed power target, as well as some subpolynomial targets. What is added here is the arbitrary threshold $f(N) \to \infty$, the quantitative fixed-target natural-counting estimate, and one explicit clock retained through that global conclusion. The raw lower bracket is a checked deterministic complement, not a priority claim. Gonçalves–Greenfeld–Madrid give the closest general almost-bounded extension of Tao's framework, but retain logarithmic-density averaging.

The exponents d_0 and $1/32000$, both clocks, and the displayed numerical inequalities are explicit. The transport prefactors $C_{\text{hit}} = 184$ and $C_{\text{tr}} = 384256$ are explicit and checked as well. The transport threshold x_0 and the resulting fixed-target prefactor are existential, so the theorem is not a practical finite-range estimate for a specified N_0.

Neither the exponent, the upper clocks, nor the transport prefactors are claimed optimal. Rhin's sharper 7.616 line [13, p. 160, (8)] is not used because its effective threshold is not printed in a form consumed by the proof. A stronger phase input with a certified finite handoff would improve d without changing the density or time arguments. The raw lower bracket rules out an $o(\log N)$ clock for the square-root target, but leaves a wide constant-factor gap.

We make no priority claim of being the first Collatz formalization without a systematic survey. The narrower, checkable claim is that the declarations in section 8 compile at the stated commit with the reported axiom footprint. The expanded transport proof above is intended to make the central argument assessable without requiring a referee to read Lean, while the appendices preserve the exact formal correspondence.

Appendix A. Technical conventions and exact source laws

This appendix records the longer finite identities and the exact formal correspondence used in the proof above. It is an integral part of the article. Lean module paths beginning `ND/`, `Tao/`, or `NumberTheory/` are relative to `Erdos1135/` in the frozen repository; explicit `scripts/` and `docs/` paths are relative to the repository root. The notation is reader-facing and does not duplicate every dependent type in the Lean API.

A.1. **Exact source blocks, bands, and passage laws.** Following Tao's exact concrete choice, set $\alpha = 1001/1000$. At a natural passage target B, the two source scales are

$$y_b \in \{B^\alpha, B^{\alpha^2}\}.$$

The corresponding inclusive odd block is

$$\{N \text{ odd} : \lceil y_b \rceil \le N \le \lfloor y_b^\alpha \rfloor\}.$$

Define

$$L_b = (\alpha - 1)\log y_b, \qquad J_b = \lfloor L_b \rfloor, \qquad \beta_b = L_b/J_b,$$
$$z_{b,j} = y_b e^{j\beta_b}, \qquad \mathcal{B}_{b,j} = \{N \text{ odd} : z_{b,j} \le N < z_{b,j+1}\}.$$

For large B, $J_b > 0$, and $z_{b,J_b} = y_b^\alpha$. Consequently the block is the disjoint union of the J_b half-open bands and the possible odd natural at the exact real endpoint y_b^α. The latter carrier has cardinality at most one; it must not be silently declared empty.

On a nonempty band define the exact flat and harmonic laws

$$\mu^{\mathrm{F}}_{b,j}(N) = \frac{1}{\#\mathcal{B}_{b,j}}, \tag{64}$$

$$\mu^{\mathrm{H}}_{b,j}(N) = \frac{N^{-1}}{\sum_{M\in\mathcal{B}_{b,j}} M^{-1}}. \tag{65}$$

They live on the same finite carrier. Their exact normalizer estimates, writing $z = z_{b,j}$ and $\mathcal{H}_{b,j} = \sum_{M\in\mathcal{B}_{b,j}} M^{-1}$, include

$$\frac{1}{\mathcal{H}_{b,j}} \le 4, \qquad \left|\frac{\beta_b}{\mathcal{H}_{b,j}} - 2\right| \le \frac{12}{z}, \qquad \frac{z}{\#\mathcal{B}_{b,j}} < 3, \qquad \left|\frac{z(e^{\beta_b}-1)}{\#\mathcal{B}_{b,j}} - 2\right| < \frac{6}{z}. \tag{66}$$

The flat density is pointwise less than e^{β_b} times the harmonic density. This one-sided comparison is used for the flat no-hit estimate; no unproved two-sided valuation comparison is needed.

For an odd source N, let $T_B(N)$ be its first Syracuse hitting time of $[1,B]$, with value ∞ on no hit. The genuine passage mass of an event $E \subseteq [1,B]$ under a source law μ is

$$P_\mu(E) = \mu\{N : T_B(N) < \infty, \mathrm{Syr}^{T_B(N)}(N) \in E\}.$$

This may have total mass less than one. Following Tao's deliberately artificial convention, the later totalized passage law puts all no-hit mass at the distinguished output 1; totalization is postponed until two genuine laws have first been compared event by event.

Appendix B. Phase gap and the strict/weak endpoint estimate

The article defines the strict and weak endpoint discrepancies in (16) and derives the common bound

$$D_N(\phi) \le \min\left\{1, \left(6 + \frac{2}{\pi c}\right) N^{-1/\kappa}\right\}.$$

For later formulas, abbreviate this exact capped majorant by

$$\overline{D}_{c,\kappa}(N) := \min\left\{1, \left(6 + \frac{2}{\pi c}\right) N^{-1/\kappa}\right\} \qquad (N \ge 1). \tag{67}$$

The checked endpoint-sensitive statement is `ndPhaseOrbit_strictWeakEndpointDiscrepancyLE` in `ND/Discrepancy/PhaseToDbar.lean`. Keeping both endpoint conventions in that one declaration is what permits half-open passage bands and inclusive finite target windows to share the later scalar majorant. The exact formal reconstruction of the Rhin input is recorded in appendix E below.

Appendix C. The common-profile estimate

This section expands the central proposition in the article. Put

$$n_0(B) = \lfloor \log_2 B/10 \rfloor, \qquad m_0(B) = \lfloor \log B/100000 \rfloor,$$

$$m_B = \left\lceil m_0(B)(\log B)^{-2/5} \right\rceil, \qquad W = C\sqrt{n_0(B)\log n_0(B)}.$$

All formulas below are used only after B is large enough that $n_0(B) > 1$, $m_0(B) > 0$, $J_b > 0$, and the displayed bands and tubes are nonempty; the finite prefix is absorbed later. Here $n_0(B)$ is the local passage horizon and tube scale, $m_0(B)$ is the exact lost-window step-back time, and m_B is the shorter affine reference depth. They enter different declarations and are not interchangeable schedules. The checked reference proof eventually takes $C = 40$. The reference level has scale $m_B \asymp (\log B)^{3/5}$, while the physical tube width has scale $W \asymp \sqrt{\log B \log\log B}$. The construction is asymptotic: $m_0(B) = 0$ for $1 \le B < e^{100000}$.

Let $p_m(r)$ be the exact iid-geometric affine law defined before (20): it is the distribution of $b(a)2^{-\sum_i a_i} \bmod 3^m$, with $\mathbb{P}(a_i = k) = 2^{-k}$ and the displayed recursion for b. Define

$$Z_m(F) = \sum_{M \in F} \frac{3^m p_m(M \bmod 3^m)}{M}.$$

Put

$$g_B = (\log B)^{7/10}, \qquad \Lambda_B = (4/3)^{m_0(B)} B, \qquad \mathcal{W}_B = \mathbb{N} \cap [e^{-g_B}\Lambda_B, e^{g_B}\Lambda_B].$$

For $E \subseteq \{1, \ldots, B\}$, the exact common target is

$$E'_B(E) = \{M \in \mathcal{W}_B : M \text{ odd},\ \mathrm{Syr}^{m_0(B)}(M) \leq B,$$
$$\mathrm{Syr}^k(M) > B\ (0 \leq k < m_0(B)),\quad \mathrm{Syr}^{m_0(B)}(M) \in E\}. \tag{68}$$

Thus M first enters $[1, B]$ at exactly $m_0(B)$. This semantics is definitionally anchored by `Tao.taoSection5EPrime` in `Tao/Section5/PassSchedule.lean`; its exact-hit predicate `Tao.syracuseFirstHitAtMost` is defined in `Tao/Syracuse/FirstPassage.lean`. The set has no source-branch parameter. The common profile is

$$\mathcal{Z}_B(E) = \frac{2}{\log(4/3)} Z_{m_B}(E'_B(E)). \tag{69}$$

The checked common-Z cap gives, eventually and uniformly in E,

$$0 \leq \mathcal{Z}_B(E) \leq 40. \tag{70}$$

It combines the checked payload-free cap $Z_{m_B}(E'_B(E)) \leq 5$ with $2/\log(4/3) \leq 8$. The declaration is `eventually_ndA5ReferenceCommonProfile_mem_Icc_forty`. This profile bound is logically independent of the equal-valued tube choice $C = 40$.

C.1. **The finite affine-to-Gaussian calculation.** Here is the finite estimate suppressed in the main article's proof of the scalar majorant. Put

$$\delta = 2 - \log_2 3 = \frac{\log(4/3)}{\log 2}.$$

For a band endpoint $z_{b,j}$ and a terminal endpoint M, define the physical affine phase

$$\mathcal{A} = \frac{\log(z_{b,j}/M)}{\log 2}.$$

The strict selector is

$$\nu \longmapsto \lfloor \mathcal{A} - \delta\nu \rfloor + 1,$$

where the added 1 is retained even at an integral phase. Write

$$x_\nu = \lfloor \mathcal{A} - \delta\nu \rfloor + 1, \qquad d_\nu = |x_\nu|, \qquad L_\nu = 2\nu + x_\nu.$$

Equivalently, since $\theta = \log_2 3 = 2 - \delta$, $L_\nu = \lfloor \mathcal{A} + \nu\theta \rfloor + 1$. This connects the signed-selector coordinates here with the raw endpoint levels used below. If $a_1, \ldots, a_\nu$ are independent positive geometric variables with $\mathbb{P}(a_i = k) = 2^{-k}$, define the previously implicit raw mass by

$$q_\nu = \mathbb{P}\left(\sum_{i=1}^{\nu} a_i = L_\nu\right), \qquad \mathcal{T}(\mathcal{A}, W) = \{\nu \in \mathbb{N} : |x_\nu| < W\}.$$

Thus $\mathcal{T}(\mathcal{A}, W)$ is the exact strict intrinsic tube used in the sums below. The physical interior, or moderate-window, guard is

$$\bigl|\log M - \bigl(\log B + m_0(B)\log(4/3)\bigr)\bigr| \leq \frac{4}{25} W. \tag{71}$$

Its logarithmic radius $(4/25)W$ is $O((\log B)^{1/2}(\log\log B)^{1/2})$, whereas the canonical lost-window radius is $g_B = (\log B)^{7/10}$. Thus the moderate window is eventually strictly contained in the lost window; the interior/exterior estimates below control the intervening region. The real center of the selector is $S = \mathcal{A}/\delta$. The exact endpoint-crossing and tube-containment inequalities give

$$S > 0, \qquad |\nu - S| \le S/3 \tag{72}$$

throughout the physical tube. This is the orientation required by the Gaussian recentering step.

For $s > 0$, write

$$\Phi_s(x) = \frac{1}{\sqrt{4\pi s}} e^{-x^2/(4s)}.$$

The signed local limit first compares the raw endpoint mass at level ν with Φ_ν, with relative error

$$\eta_\nu = 16\left(\frac{d_\nu}{\nu} + \frac{d_\nu^3}{\nu^2}\right), \qquad \epsilon_\nu = \frac{1}{2\nu+1} + 2\eta_\nu.$$

The eventual physical tube used here has $\nu > 0$, so these denominators are nonzero. Recentering from ν to S uses the exact factorization

$$\Phi_\nu(x) = \Phi_S(x) e^{e(\nu,S,x)},$$

and the bound

$$|e(\nu, S, x)| \le \zeta_\nu := \frac{3}{4}\frac{|\nu - S|}{S} + \frac{3}{8}x^2\frac{|\nu - S|}{S^2}.$$

Under (72) and $\zeta_\nu \le 1$, this yields

$$|\Phi_\nu(x) - \Phi_S(x)| \le 2\Phi_S(x)\zeta_\nu, \qquad \Phi_\nu(x) \le \Phi_S(x)(1 + 2\zeta_\nu).$$

Combining the two comparisons preserves the cross term:

$$|q_\nu - \Phi_S(x_\nu)| \le \Phi_S(x_\nu)(\epsilon_\nu(1 + 2\zeta_\nu) + 2\zeta_\nu). \tag{73}$$

Replacing $1 + 2\zeta_\nu$ prematurely by a constant would still prove decay, but would lose the checked source-level budget.

Set

$$R = \lceil W \rceil - 1, \qquad \mathrm{lo} = \max\left\{0, \left\lfloor \frac{\mathcal{A} - R}{\delta} \right\rfloor + 1\right\}.$$

Thus lo is the natural lower endpoint of the strict physical tube; a negative signed endpoint is clipped to zero. In the eventual moderate-window regime, the checked geometry gives $\mathrm{lo} > 0$. Define

$$G_{\mathrm{raw}} = \sum_{\nu \in \mathcal{T}(\mathcal{A},W)} q_\nu, \qquad G_{\mathrm{gauss}} = \sum_{\nu \in \mathcal{T}(\mathcal{A},W)} \Phi_S(x_\nu).$$

Summing (73) over that tube gives a relative comparison $|G_{\mathrm{raw}} - G_{\mathrm{gauss}}| \le \mathcal{E}_{\mathrm{loc}} G_{\mathrm{gauss}}$, where

$$\mathcal{E}_{\mathrm{loc}} = 44\frac{W}{\mathrm{lo}} + 42\frac{W^3}{\mathrm{lo}^2}. \tag{74}$$

No cardinality-times-supremum estimate is used. The strict-multiplicity Gaussian quadrature gives

$$|G_{\mathrm{gauss}} - 1/\delta| \le \mathcal{Q},$$
$$\mathcal{Q} = \left(2 + \frac{1}{\delta}\right)\Phi_S(0) + \frac{4S}{\delta R}\Phi_S(R). \tag{75}$$

The term $2\Phi_S(0)$ is the strict-multiplicity error and is not multiplied by $1/\delta$. Since $G_{\mathrm{gauss}} \le 1/\delta + \mathcal{Q}$, the exact composition is

$$|G_{\mathrm{raw}} - 1/\delta| \le \mathcal{E}_{\mathrm{loc}}(1/\delta + \mathcal{Q}) + \mathcal{Q}. \tag{76}$$

The physical lower-endpoint bounds $13.74 \le n_0(B)/\mathrm{lo} \le 42.44$, together with the moderate-window guard, imply

$$\mathcal{Q} \le \frac{42}{\sqrt{n_0(B)}}, \qquad \mathcal{E}_{\mathrm{loc}} \le 76000(C + C^3)\frac{(\log n_0(B))^{3/2}}{\sqrt{n_0(B)}}.$$

In particular (76) is bounded by

$$3650000(C + C^3)\frac{(\log n_0(B))^{3/2}}{\sqrt{n_0(B)}}.$$

The independent phase-discrepancy/Koksma term contributes the first line of (84) below. This separates the local Gaussian rate from the Diophantine rate and makes clear why only the latter controls the final exponent.

C.2. **Two normalized physical profiles.** The same original endpoint M and the same positive physical valuation window are used for both laws. Write $z = z_{b,j}$, $A = \log(z/M)/\log 2$, $\theta = \log_2 3$, and

$$\mathcal{I}_{b,j} = \{\nu \in \mathbb{N}_{>0} : \nu + m_0(B) \in [\lceil \log(z/B)/\log(4/3) - 3W \rceil , \\ \left\lfloor \log(ze^{\beta_b}/B)/\log(4/3) + 3W \right\rfloor]\}.$$

For any real A', let

$$q(A', W; \nu) = \mathbf{1}_{\{\nu \in \mathcal{T}(A',W)\}} \mathbb{P}\left(\sum_{i=1}^{\nu} a_i = \lfloor A' + \nu\theta \rfloor + 1\right), \qquad \mathbb{P}(a_i = k) = 2^{-k}.$$

For $r \in \{0, 1\}$, put

$$L_{\nu,r} = \lfloor A + r + \nu\theta \rfloor + 1, \qquad u_{\nu,r} = (L_{\nu,r} - (A + \nu\theta)) \log 2,$$

so $L_{\nu,0} = L_\nu$ and $q(A, W; \nu) = q_\nu$; the row $r = 1$ is the literal first shift, not a replacement of the base mass. Define

$$Q_{\nu,r} = \mathbf{1}_{\{0<u_{\nu,r}<\beta_b\}} q(A + r, W; \nu).$$

The two literal unnormalized sums are

$$R_H = \sum_{\nu \in \mathcal{I}_{b,j}} \sum_{r=0}^{1} Q_{\nu,r}, \tag{77}$$

$$R_F = \sum_{\nu \in \mathcal{I}_{b,j}} \sum_{r=0}^{1} Q_{\nu,r} \exp(((r + 1) - \{A + \nu\theta\}) \log 2) . \tag{78}$$

Thus the flat exponential is attached exactly once, and zero extension from the physical window to the two intrinsic tubes changes neither sum.

The raw errors are also explicit. The interior guards ensure that $I_0 := \mathcal{T}(A, W)$ is nonempty and that $\ell := \mathrm{lo} > 0$. With these domain facts, let

$$I_0 = \mathcal{T}(A, W), \quad \ell = \mathrm{lo}, \quad Z_0 = \sum_{\nu \in I_0} q(A, W; \nu), \quad D = \overline{D}_{c,\kappa}(\#I_0),$$

$$\varepsilon_6 = 3650000(C + C^3)\frac{(\log n_0(B))^{3/2}}{\sqrt{n_0(B)}}, \quad G = \frac{64(\#I_0)W}{\ell}, \quad S_0 = 2\frac{W}{\ell}Z_0 + \frac{24}{\sqrt{n_0(B)}}.$$

Define

$$E_H^{\mathrm{raw}} = Z_0 DG + S_0 + \frac{\beta_b}{\log 2}\varepsilon_6, \tag{79}$$

$$E_F^{\mathrm{raw}} = Z_0(2e^{\beta_b} - 1)DG + e^{\beta_b} S_0 + \frac{e^{\beta_b} - 1}{\log 2}\varepsilon_6. \tag{80}$$

The finite affine-to-Gaussian calculation proves

$$|R_H - \beta_b/\log(4/3)| \le E_H^{\mathrm{raw}}, \qquad |R_F - (e^{\beta_b} - 1)/\log(4/3)| \le E_F^{\mathrm{raw}}. \tag{81}$$

In `ND/Discrepancy/`, equations (77)–(80) unfold `ndA5ReciprocalTwoProfileA6Error` and `ndA5FlatTwoProfileA6Error` in `A5TwoProfilePhysicalA6.lean`; their common bound is proved in `A5ReferenceInteriorRate.lean`. With the exact normalizers from (66),

$$\widehat{R}_H = R_H \Bigg/ \sum_{N\in\mathcal{B}_{b,j}} N^{-1}, \qquad \widehat{R}_F = \frac{zR_F}{\#\mathcal{B}_{b,j}},$$

the checked scalar bridge is

$$\left|\widehat{R}_H - \frac{2}{\log(4/3)}\right| \le 4E_H^{\mathrm{raw}} + \frac{12}{z\log(4/3)} \le \mathcal{E}_{B,C}, \tag{82}$$

$$\left|\widehat{R}_F - \frac{2}{\log(4/3)}\right| \le 3E_F^{\mathrm{raw}} + \frac{6}{z\log(4/3)} \le \mathcal{E}_{B,C}. \tag{83}$$

The final two inequalities are exactly `ndA5ReferenceInteriorA6Errors_le_scalarMajorant`; no asymptotic normalizer is substituted.

The physical tube has cardinality between W and $7W$. Finite Abel summation, the endpoint discrepancy (17), the signed local limit, and the native Gaussian quadrature estimate produce the common scalar majorant

$$\begin{aligned}\mathcal{E}_{B,C} = {}& \frac{39927552}{25}\left(6+\frac{2}{\pi c}\right)C^2(\log n_0(B))W^{-1/\kappa}\\ &+\left(\frac{101856}{25}C + 288 + 54750000(C+C^3)\right)\frac{(\log n_0(B))^{3/2}}{\sqrt{n_0(B)}} + \frac{12}{B\log(4/3)}.\end{aligned} \tag{84}$$

The first summand is the phase term. Since $n_0(B) \asymp \log B$,

$$\begin{aligned}(\log n_0(B))W^{-1/\kappa} &= C^{-1/\kappa} n_0(B)^{-1/(2\kappa)}(\log n_0(B))^{1-1/(2\kappa)}\\ &= O\left((\log B)^{-1/(2\kappa)}(\log\log B)^{1-1/(2\kappa)}\right).\end{aligned}$$

Therefore, for every $d < 1/(2\kappa)$, it is $o((\log B)^{-d})$. The physical-rate term has size

$$O\big((\log B)^{-1/2}(\log\log B)^{3/2}\big),$$

and the last term is smaller again. The terminal approximation contributes powers

$$1/20, \quad 3/10, \quad 9/5, \quad 6, \quad 6/5;$$

hence $d < 1/20$ is its strict binding guard.

C.3. **Interior, exterior, and terminal reconstruction.** For an odd source N, put

$$a_i(N) = \nu_2(3\,\mathrm{Syr}^{i-1}(N)+1), \qquad \mathcal{G}_{B,C} = \left\{N : \left|\sum_{i=1}^{k} a_i(N) - 2k\right| < W \text{ for every } k \le n_0(B)\right\}.$$

This is the full-prefix valuation-tube event called `FullGood` in the Lean modules. Terminal coverage restricts the first-passage reconstruction to $\mathcal{G}_{B,C}$. "Nominal terminal mass" means the ideal Geom(2)/affine endpoint weight on the same finite terminal carrier; no endpoint or event is changed when the exact mass is replaced by that weight.

Multiplying (82) and (83) by the common nonnegative kernel and summing over interior endpoints loses no endpoint-cardinality factor. At the common reference level, the whole target $E'_B(E)$ has a fixed cap, yielding five copies of

$$R_d(B) = (\log B)^{-d}$$

for the interior profile. The exterior profile has logarithmic-cube decay. Charging both that profile and the omitted exterior copy of the center costs one additional $R_d(B)$. Thus both full physical reference profiles are within $6R_d(B)$ of $\mathcal{Z}_B(E)$.

The remaining comparison to genuine first passage has the following exact budget.

Passage reconstruction item	Cost in units of $R_d(B)$
Full-prefix tube/terminal coverage	$1/4$
Exact-to-nominal terminal masses	$1/2$
Aggregation at the common reference level	$1/4$
Interior common center	5
Exterior profile and omitted center	1
Total	7

Consequently, for either branch, either band law, and every output event,

$$\left|P^{\sigma}_{b,j}(E) - \mathcal{Z}_B(E)\right| \le 7R_d(B). \tag{85}$$

The quantifier order is important: one eventual threshold works uniformly in the branch, every band $0 \le j < J_b$, endpoint event, and the flat/harmonic choice. For traceability, terminal coverage is supplied by the two declarations in `ND/Discrepancy/A5BandCoverageRate.lean`; exact-to-nominal replacement has raw bound $11B^{-9/10}$; and `eventually_abs_ndA5ReferenceTerminalNominalSums_sub_physicalProfiles_le_rate` supplies the final quarter. Their respective absorptions are $1/4, 1/2, 1/4$, exactly as in the table.

Appendix D. Whole-source, totalization, and real-floor ledgers

Let $P^{\sigma}_b(E)$ be the genuine passage mass from the whole source law. The exact source disintegration is

$$P^{\sigma}_b(E) = \sum_{j<J_b} w^{\sigma}_{b,j} P^{\sigma}_{b,j}(E) + T^{\sigma}_b(E), \tag{86}$$

where

$$\sum_{j<J_b} w^{\sigma}_{b,j} + t^{\sigma}_b = 1, \qquad 0 \le T^{\sigma}_b(E) \le t^{\sigma}_b, \qquad t^{\sigma}_b \le 4/B. \tag{87}$$

The formulas hold separately for exact counting numerators and exact reciprocal-mass numerators; denominator normalization alone would not imply the event identity (86).

Using (85), (70), and (87),

$$\begin{aligned} |P^{\sigma}_b(E) - \mathcal{Z}_B(E)| &\le 7R_d(B) + 40t^{\sigma}_b \\ &\le 7R_d(B) + 160/B \\ &\le 8R_d(B) \end{aligned}$$

eventually. Exact weight normalization eliminates a band-count loss. The triangle inequality gives $16R_d(B)$ for same-branch flat/harmonic passage and for cross-branch harmonic/harmonic passage.

Totalize at the one common finite output carrier $\{M \in \mathbb{N} : M \le B\}$, using 1 as the no-hit sentinel. For an event avoiding 1, totalized mass equals genuine mass. For an event containing 1, apply the same event estimate to its complement inside the finite carrier. A finite Hahn decomposition is then used exactly once:

$$\sup_E |\mu(E) - \nu(E)| \le 16R_d(B) \quad \Longrightarrow \quad \|\mu - \nu\|_1 \le 32R_d(B). \tag{88}$$

Now let x be real and $B = \lfloor x \rfloor$. The exact floor-source errors are

$$\delta_U = 96x^{-99/100}, \qquad \delta_H = \frac{96000}{B\log B}.$$

The checked raw rows are

$$\text{uniform no-hit} \le 44B^{-1/32000} + \delta_U, \tag{89}$$

$$\text{same-branch full } \ell^1 \le 2\delta_U + 32R_d(B) + 2\delta_H. \tag{90}$$

The passage triangle is exactly

$$U_{\mathrm{real}} \longrightarrow U_{\mathrm{floor}} \longrightarrow H_{\mathrm{floor}} \longrightarrow H_{\mathrm{real}}.$$

For large x, the elementary bounds $x/2 \le B$ and $(\log x)/2 \le \log B$ yield the final ledger

$$44B^{-1/32000} + 96x^{-99/100} \le (88 + 96)x^{-1/32000},$$

$$2(96x^{-99/100}) + 32(\log B)^{-d} + \frac{192000}{B \log B} \le (192 + 64 + 384000)(\log x)^{-d}.$$

Thus the stated two-block theorem may use

$$C_{\mathrm{hit}} = 184, \qquad C_{\mathrm{tr}} = 384256.$$

The no-hit exponent $1/32000$ remains separate from d; replacing both by their minimum would destroy the sharper transport exponent.

For $\kappa = 143/10$, the guards are

$$0 < d < 1/20, \qquad d < 1/(2\kappa) = 5/143.$$

Because $5/143 < 1/20$, the phase guard is binding. This is the complete origin of the open cap in the main theorem.

Appendix E. Formal reconstruction of the Rhin input

The conventional proof in the article invokes Rhin's printed proposition [13]. The Lean proof instead reconstructs the weaker eventual large-height statement actually needed for the phase gap. This section fixes the source/certificate boundary.

The registered primary-source PDF has SHA-256

`1ada4c7f51933af33feb14757e4fcd46c19f33748da93d950b8e6a9b647cfd1e`.

The load-bearing attribution is:

Source location	Data used
Rhin p. 159	Termwise coefficient/integral construction.
Rhin p. 160, (7)	Height, integer linear form, and exponent-13.3 theorem surface; the formal target is a weaker eventual large-height form.
Rhin p. 162	Six displayed factors, fixed scalar 12^7, and $Q_6(X) = 19X^2 - 108X + 144$. The formal certificate uses independently checked millionth rationals matching the displayed decimals; it attributes no additional precision to the source.

The square shifts, ordered-moment determinant, determinant threshold 8, continuous 38-cell remainder bound, prefactored base-three LCM estimate, and grouped exponent 13 are derived exact certificates. They are not statements attributed to Rhin. The proof does not substitute the later -104 polynomial appearing in other work.

The dependency-ordered formal layers, inside `Erdos1135/NumberTheory/Rhin/` unless another path is shown, are:

(1) `Statement.lean`, `RangeLCM.lean`, `LiteralPolynomial.lean`, and `RemainderCore.lean` freeze the statement surface and source-independent arithmetic.
(2) `Irrationality.lean`, `LcmGrowth.lean`, `LiteralSupport.lean`, and `LiteralRemainder.lean` prove nonvanishing, LCM growth, support, and the literal-to-generic remainder bridge.
(3) `LiteralCoefficientBound.lean`, generated `RemainderCells.lean`, and `ContinuousRemainder.lean` supply the finite coefficient and continuous interval certificates.
(4) `LargeHeight.lean` gives the termwise row construction, integrality, determinant, and grouped transference argument.
(5) `FiniteAbsorption.lean` absorbs the remaining finite height range symbolically. `LiteralLargeHeight.lean` joins the concrete rows and LCM bound to prove `literalRhinLargeHeightBound`.

(6) `ND/RhinPhaseGap.lean` is the sole ND-facing adapter and proves `existsPhaseGapRhin`.

The finite absorption uses positivity of every nearest-integer gap and a finite minimum, capped by 1. It begins at frequency $q = 1$ and never evaluates a historical low-frequency table. The formal milestone therefore does not claim an explicit phase constant, a printed crossover, Rhin's full constant-one all-height theorem, or the sharper exponent 7.616.

The sole generated Lean certificate in this Rhin reconstruction is reproducible by

```
python scripts/generate_rhin_remainder_cells.py \
  --output /tmp/RemainderCells.replay.lean
cmp /tmp/RemainderCells.replay.lean \
  Erdos1135/NumberTheory/Rhin/RemainderCells.lean
sha256sum /tmp/RemainderCells.replay.lean \
  Erdos1135/NumberTheory/Rhin/RemainderCells.lean
```

The checked file has size 37204 bytes and SHA-256

`b7a0854e378a8a519f411bf4b3fd3693bf7b4454f472d1195ee2fcfd3b231e01`.

The Python generator is not proof evidence. Its output contains exact rational data and ordinary theorem shells; Lean's kernel checks the cells by exact reduction.

Appendix F. Lean theorem map, trust boundary, and reproduction

The primary paper import is

```
import Erdos1135.ND.LogTime.Paper
```

The paper's theorem-bearing surfaces map as follows. Each entry gives an exported declaration followed by its source module.

(1) **Tao precursor endpoint:** `taoAlmostBoundedColMin_checked`; `Tao/AlmostBounded.lean`.

(2) **Rhin large height:** `literalRhinLargeHeightBound`; `NumberTheory/Rhin/LiteralLargeHeight.lean`.

(3) **Global phase gap:** `existsPhaseGapRhin`; `ND/RhinPhaseGap.lean`.

(4) **Strict/weak phase discrepancy:** `ndPhaseOrbit_strictWeakEndpointDiscrepancyLE`; `ND/Discrepancy/PhaseToDbar.lean`.

(5) **Raw error definitions and normalized flat/harmonic center:** `ndA5ReciprocalTwoProfileA6Error`, `ndA5FlatTwoProfileA6Error`, and `eventually_ndA5TwoProfileNormalizedA6`; `ND/Discrepancy/A5TwoProfilePhysicalA6.lean` and `ND/Discrepancy/A5TwoProfileNormalizedA6.lean`.

(6) **Exact common target and exact first hit:** `taoSection5EPrime`, `mem_taoSection5EPrime_iff`, and `syracuseFirstHitAtMost`; `Tao/Section5/PassSchedule.lean` and `Tao/Syracuse/FirstPassage.lean`.

(7) **Common-profile cap $0 \le \mathcal{Z}_B(E) \le 40$:** `eventually_ndA5ReferenceCommonProfile_mem_Icc_forty`; `ND/Discrepancy/A5ReferenceFullCenterRate.lean`.

(8) **Interior scalar rate and factor five:** `eventually_ndA5ReferenceInteriorDecay_referenceLevel_le_five`; `ND/Discrepancy/A5ReferenceInteriorRate.lean`.

(9) **Full factor-six common center:** `eventually_ndA5ReferenceFullPhysicalDecay_referenceLevel_le_six`; `ND/Discrepancy/A5ReferenceFullCenterRate.lean`.

(10) **Genuine per-band factor seven:** `eventually_ndA5GenuineBandPassDecay_referenceLevel_le_seven`; `ND/Discrepancy/A5ReferenceGenuineBandRate.lean`.

(11) **Terminal aggregation quarter:** `eventually_abs_ndA5ReferenceTerminalNominalSums_sub_physicalProfiles_le_rate`; `ND/Discrepancy/A5ReferenceTerminalAggregateRate.lean`.

(12) **Exact whole-source event mixture:** `ndA5SourcePMF_eventMass_eq_band_mixtures_add_top`; `ND/Band/A5SourceMixture.lean`.

(13) **Whole-source factor eight:** `eventually_ndA5GenuineSourcePassDecay_referenceLevel_le_eight`; `ND/Discrepancy/A5ReferenceGenuineSourceRate.lean`.

(14) **Full-ℓ^1 factor thirty-two:** `eventually_passFullL1_natCast_taoSection5SourceY_le_thirty_two`; `ND/Discrepancy/A5ReferenceTotalizedSourceRate.lean`.

(15) **Real-floor raw ledger:** `eventually_ndA5RealSameBranchRawBounds`; `ND/Discrepancy/A5ReferenceRealFloorPass.lean`.

(16) **Scheduled no-hit bounds $44B^{-1/32000}$ and $184x^{-1/32000}$:** `eventually_ndA5FlatSourceNoHitWithinMass_le_forty_four`, `eventually_uniformScheduledNoHitProbability_natCast_sourceY_le`, and `eventually_uniformScheduledNoHitProbability_transportSourceY_le`; `ND/LogTime/UniformScheduledNoHit.lean`.

(17) **Same-d transport theorem:** `ndA5ReferenceNDM2Bounds_sameD`; `ND/Discrepancy/A5ReferenceNDM2.lean`.

(18) **Timed geometric trace:** `exists_ndA5ReferenceSection3AmbientTimeTerminal`; `ND/LogTime/A5Section3Time.lean`.

(19) **All-d Rhin rate and exact specialization:** `ndRhinLogTimeRate_sameD` and `ndRhinLogTimeRate_sameD_6993_200000`; `ND/LogTime/RhinUnconditional.lean`.

(20) **All-d fixed-target bound:** `ndRhinND31LogTimeBounds_sameD`; `ND/LogTime/ND31.lean`.

(21) **Clock formulas and numerical bounds:** `ndSyracuseLogTimeConstant_eq_501501_div`, `ndSyracuseLogTimeConstant_lt_145`, `ndCollatzLogTimeConstant_eq_1509503_div`, and `ndCollatzLogTimeConstant_lt_436`; `ND/LogTime/Paper.lean`.

(22) **Exact raw value identity:** `collatz_iterate_powTwo_mul_syracuseValuationTime`; `Tao/Syracuse/LogTimeCollatzBridge.lean`.

(23) **Exact dyadic logarithm identity:** `powTwo_add_log_div_eq_log_mul_div`; `Tao/Syracuse/LogTimeCollatzBridge.lean`.

(24) **Fixed-target three-contribution bridge:** `pmfProb_fixedTargetBadWithin_add_le`; `ND/LogTime/FixedTargetBridge.lean`.

(25) **Timed-to-untimed implication:** `ndChainStatement_of_logTimeChainStatement`; `ND/LogTime/Paper.lean`.

(26) **Bundled paper theorem:** `ndRhinLogTimePaperPackage`; `ND/LogTime/Paper.lean`.

(27) **Pointwise raw lower clocks:** `Tao.collatz_hit_time_log_lower` and `Tao.collatz_sqrt_hit_time_lower`; `Tao/Syracuse/RawTimeLowerBound.lean`.

(28) **Raw square-root bracket:** `ndRhinRawCollatzSqrtLogTimeBracket`; `ND/LogTime/Paper.lean`.

The commit pin identifies the complete source tree.

The project uses Lean `v4.30.0-rc2` [10] and Mathlib commit `5450b53e5ddc75d46418fabb605edbf36bd0beb6` [11]. At the exact frozen commit, a fresh clone built the complete paper import and `leanchecker -verbose` replayed it with exit status zero. The complete release source and replay files are available from the ProofAtlas formalization page. From the root of the distributed source tree, a minimal reproduction sequence is

```
lake exe cache get
lake build Erdos1135.ND.LogTime.Paper
lake env leanchecker --verbose Erdos1135.ND.LogTime.Paper
```

The distributed source includes its committed `lake-manifest.json`; do not run `lake update`, which would re-resolve and may overwrite the pinned dependency graph.

For the declarations cited in the article, `#print axioms` reports exactly

`propext`, `Classical.choice`, `Quot.sound`.

Scoped audits found no `sorry`, `admit`, custom axiom, `unsafe`, `native_decide`, or dependency on the project's open Collatz-conjecture placeholder in this theorem cone. `leanchecker` replays declarations with Lean's checker and detects environment inconsistencies; it is not an independent theorem prover.

Acknowledgments. The author thanks the developers of Lean and Mathlib and the authors of the primary mathematical sources used in the proof.

References

[1] J.-P. Allouche, *Sur la conjecture de "Syracuse–Kakutani–Collatz"*, Séminaire de Théorie des Nombres, 1978–1979, Exp. No. 9, 15 pp., CNRS, Talence, 1979.

[2] J.-P. Allouche, *T. Tao and the Syracuse conjecture*, EMS Magazine **123** (2022), 12–16. doi:10.4171/MAG-64.

[3] C. J. Everett, *Iteration of the number-theoretic function* $f(2n) = n$, $f(2n + 1) = 3n + 2$, Adv. Math. **25** (1977), no. 1, 42–45. doi:10.1016/0001-8708(77)90087-1.

[4] F. Gonçalves, R. Greenfeld, and J. Madrid, *Generalized Collatz maps with almost bounded orbits*, Indiana Univ. Math. J. **74** (2025), no. 1, 1–46. doi:10.1512/iumj.2025.74.60034; arXiv:2111.06170.

[5] E. Heppner, *Eine Bemerkung zum Hasse–Syracuse-Algorithmus*, Arch. Math. (Basel) **31** (1978), 317–320. doi:10.1007/BF01226454.

[6] M. Inselmann, *An approximation of the Collatz map and a lower bound for the average total stopping time*, arXiv:2402.03276v3, last revised 13 August 2024. arXiv:2402.03276v3.

[7] A. S. Kalafatelis, *Uniform-window shell kernels for the Syracuse map: renewal structure and a primitive-frequency obstruction*, Preprints.org 2026, 202604.0411 (v1). doi:10.20944/preprints202604.0411.v1.

[8] I. Korec, *A density estimate for the* $3x + 1$ *problem*, Math. Slovaca **44** (1994), no. 1, 85–89. DML-CZ:133225.

[9] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Pure and Applied Mathematics, Wiley-Interscience, New York, 1974.

[10] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction—CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, 625–635. doi:10.1007/978-3-030-79876-5_37.

[11] The mathlib Community, *The Lean mathematical library*, Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs, 2020, 367–381. doi:10.1145/3372885.3373824.

[12] H. Möller, *Über Hasses Verallgemeinerung des Syracuse-Algorithmus (Kakutanis Problem)*, Acta Arith. **34** (1978), 219–226. doi:10.4064/aa-34-3-219-226.

[13] G. Rhin, *Approximants de Padé et mesures effectives d'irrationalité*, in: Séminaire de Théorie des Nombres, Paris 1985–86, Progress in Mathematics **71**, Birkhäuser, 1987, 155–164. doi:10.1007/978-1-4757-4267-1_11.

[14] J. L. Simons and B. M. M. de Weger, *Theoretical and computational bounds for m-cycles of the* $3n + 1$*-problem*, Acta Arith. **117** (2005), no. 1, 51–70. doi:10.4064/aa117-1-3.

[15] T. Tao, *Almost all orbits of the Collatz map attain almost bounded values*, Forum Math. Pi **10** (2022), e12. doi:10.1017/fmp.2022.8; arXiv:1909.03562v7, revised 16 July 2026.

[16] R. Terras, *A stopping time problem on the positive integers*, Acta Arith. **30** (1976), 241–252. doi:10.4064/aa-30-3-241-252.

Email address: 2022melodies@gmail.com